Robust discrete-time nonlinear sliding mode state estimation of uncertain nonlinear systems

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SUMMARY

In this paper, we propose a discrete-time nonlinear sliding mode observer for state and unknown input estimations of a class of single-input/single-output nonlinear uncertain systems. The uncertainties are characterized by a state-dependent vector and a scalar disturbance/unknown input. The discrete-time model is derived through Taylor series expansion together with nonlinear state transformation. A design methodology that combines the discrete-time sliding mode (DSM) and a nonlinear observer design is adopted, and a strategy is developed to guarantee the convergence of the estimation error to a bound within the specified boundary layer. A relation between sliding mode gain and boundary layer is established for the existence of DSM, and the estimation is made robust to external disturbances and uncertainties. The unknown input or disturbance can also be estimated through the sliding mode. The conditions for the asymptotical stability of the estimation error are analysed. Application to a bioreactor is given and the simulation results demonstrate the effectiveness of the proposed scheme. Copyright © 2006 John Wiley & Sons, Ltd.

1. INTRODUCTION

Over the last decade, state estimation of nonlinear systems has been an active field of research. Nonlinear observers that are based on geometric techniques had yielded significant results [1, 2], and a number of linearization methods have been proposed to handle nonlinear dynamics. For example, the results in [3–5] provide a good introduction to the geometric methods in
continuous-time domain by exact feedback linearization for nonlinear systems, which involves a state transformation through Lie derivatives. In recent years, state estimators for nonlinear systems that can effectively handle model and measurement uncertainties [6–8] have also been developed. There were also significant progresses in the implementation of these techniques in the area of process control, especially for some special classes of nonlinear systems [9, 10]. To date, most of the existing nonlinear observers design methods [11–13] and the analyses of nonlinear observers are mostly in the continuous-time domain as exact discretization of a nonlinear process model is difficult.

There are available generic methods for sliding mode observer (SMO) design, for continuous-time linear and nonlinear systems, based on the equivalent control concept for handling disturbances and modelling uncertainties [7, 8, 14–16]. In the same context, discrete-time sliding mode control (DSMC) methods have also been developed for linear and nonlinear systems [14, 17–20]. However, discrete-time sliding mode-based observer (DSMO) design has not received much attention, especially for nonlinear systems. Discrete-time sliding mode observer design for linear systems was given in [14]. In [21], the concept of sliding lattice for discrete-time systems was introduced and DSMO was designed for single-input single-output (SISO) linear systems using the Lyapunov min–max method.

In this paper, we shall design a robust discrete-time sliding mode (DSM) nonlinear estimator. We perform a discrete-time state transformation on the discrete-time plant model obtained via Taylor series expansion, and then design a discrete-time state estimator in the transformed domain. The obtained discrete-time plant dynamic model ‘copies’ the first-order dynamics of the nonlinear model. Through a nonlinear change of co-ordinates, linearization is achieved by output and output derivative injections. To overcome the problem of disturbances/uncertainties in estimation, we propose a nonlinear observer with the sliding mode approach to effectively handle the uncertainties. The uncertainties are assumed to be characterized by a bounded unknown input multiplied with a known state-dependent vector. The current state estimation technique [22] was employed in the design of estimator. Under a global Lipschitz condition for the nonlinear part, the nonlinear observers are designed under some structural assumptions such that the system is observable with respect to any control input [23]. We combined the discretized observer with SMO, which is a robust term introduced to handle the parametric uncertainties and disturbances. The disturbances in the system are also estimated through the sliding mode term.

The estimator design analysis is performed in the transformed system obtained through co-ordinate mapping by a diffeomorphism. Through an inverse state transformation, we readily obtain the estimator for the original system. It is shown that a proper design of the feedback estimation gain will ensure the ultimate boundedness of the estimation error. The ultimate bound becomes smaller with a higher sampling rate, and the quality of convergence also increases as DSM approximates the continuous-sliding mode. The estimation performance is significantly improved with the incorporation of the sliding mode term.

So far, most of the analyses of DSM have been done outside the boundary layer. In [24], the design of DSM observer for SISO nonlinear systems was presented and the existence of DSM was also established. The boundary layer is generally used to avoid chattering in standard sliding mode design. In this paper, we show that the selection of a proper boundary layer forms the basis for the existence of DSM. The relation between the sliding mode gain and boundary layer thickness is explored and chattering is completely eliminated. Previous analysis of SMO depends on equivalent control [7], and the gain is not known a priori although the disturbance
bounds are known. In our analysis, the gain and the boundary layer can be selected based on disturbance bounds.

As an application, we study the state estimation of a bioreactor in the presence of uncertainties. A good estimation accuracy is obtained by using our proposed approach. The simulation results show good performance in the presence of uncertainties.

The rest of the paper is organized as follows: Section 2 presents the main idea of the DSM phenomenon. Section 3 discusses the discrete-time estimator which includes preliminaries on discrete-time nonlinear state transformation. Section 4 contains the main result of nonlinear discrete-time observer design, the relevant analysis, existence of DSM, unknown input estimation, and the design of DSMO. Section 5 presents the application to a bioreactor with simulation results to demonstrate the effectiveness of the proposed approach. Section 6 concludes the paper.

2. EXISTENCE OF DISCRETE-TIME SLIDING MODE

When uncertainties and disturbances are present in the system, DSM cannot assure invariance or asymptotic convergence of the estimation error. The nature and existence of DSM have been discussed in the literature [14, 17–20]. The convergence of error to a final ultimate bound, where the system overcomes the disturbance, will be called the DSM. For the convergent switching to occur and the DSM to exist, the following two conditions are to be satisfied [17, 18, 20]:

\[ [s(k+1) - s(k)] \text{sign}(s(k)) < 0 \]  
\[ [s(k+1) + s(k)] \text{sign}(s(k)) > 0 \]

Here, \( s(k) \) is the sliding trajectory and \( s(k) = 0 \) is the sliding manifold. Condition (1) ensures switched motion about the hyperplane or convergence to the hyperplane if switching does not occur on the sliding manifold. Condition (2) is required to ensure convergence onto the hyperplane if switching occurs. Condition (2) is not necessary if switching does not occur on the sliding manifold. Based on the available conditions, we propose a new strategy to develop chattering-free DSM, where the system trajectory enters the boundary layer in the vicinity of the sliding mode and stays inside it forever.

2.1. Boundary layer and chattering-Free DSM

Our main idea to establish the DSM is to avoid switching on the sliding manifold. The trajectory is made to reach the sliding manifold, and once it enters the boundary layer, our estimator ensures that it is retained inside the boundary layer forever as illustrated in Figure 1. The proposed methodology can be described in two steps:

1. The sliding mode gain is designed such that the trajectory is directed towards the sliding surface, and to reach the boundary layer in finite time and maintains the sliding mode thereafter. The trajectory satisfies both conditions (1) and (2) outside the boundary layer.

2. An appropriate selection of the boundary layer and the sliding mode gain constrains the trajectory to the pre-defined boundary layer. The trajectory never crosses the otherside of boundary layer.
The confinement of the trajectory to the boundary layer, after the trajectory enters the boundary layer, prevents chattering. This is crucial for the existence of DSM. In this paper, we achieve DSM through the following saturation function:

\[
\text{sat}(\cdot, \varepsilon) = \begin{cases} 
\cdot/\varepsilon & \text{if } |\cdot| \leq \varepsilon, \\
\text{sign}(\cdot) & \text{if } |\cdot| > \varepsilon
\end{cases}
\] (3)

In standard sliding mode phenomenon, a boundary layer is generally used to avoid excessive chattering over the sliding manifold in the continuous-time domain. Inside the boundary layer, the switching function can be approximated by a linear feedback gain [25, 26]. Existing analysis of the DSM are performed outside the boundary layer and no analytical methods are available for the design of boundary layer.

We shall show that the boundary layer forms an integral part of the observer design in our proposed design methodology. The convergence of trajectory to a ultimate bound inside the boundary is considered as DSM and its existence is based on the boundary layer. Further, an analytical solution for the design of sliding mode gain and the boundary layer thickness that completely eliminates the chattering and guarantees the existence of DSM will be developed in this paper. In our analysis, we show that the boundary layer forms the basis for the existence of DSM. The boundedness of the trajectory within the boundary layer of the sliding manifold after the trajectory enters the boundary layer is also established. A similar design for existence of DSM with boundary layer was adopted in [24] for discrete-time nonlinear systems.

3. PRELIMINARIES

We consider SISO nonlinear continuous-time systems that are described by

\[
\dot{x}(t) = f(x) + b(x)u(t) + p(x)d(x, u, t)
\]

\[
y(t) = h(x)
\] (4)
where \( x = [x_1, x_2, \ldots, x_n]^T \) is the state vector comprising all the key process variables of the system, \( u \) is the control input, \( p \) is a known state-dependent distribution vector, \( y \) is the system output, \( f, b \) and \( h \) are system functions and are all nonlinear functions of the states \( x \), and \( d \) is the scalar-valued disturbance term or unknown input that accounts for modelling uncertainties and disturbance. We assume the disturbance input to be bounded as \( |d(t)| \leq d \).

### 3.1. Discrete-time plant dynamics

As the functions \( f(\cdot) \) and \( b(\cdot) \) are nonlinear in nature, the ordinary differential equation (ODE) (4) cannot be solved exactly, and hence the exact form of the discrete-time difference equation is difficult to obtain. To overcome this problem we make use of Taylor’s series expansion to discretize the system.

Denoting the discrete-time index (\( k \)) as the variable at time \( t = kT_s \), where \( T_s \) is the sampling period, we use the following Taylor series expansion of \( x \) about the variable ‘\( k \)’:

\[
x(k + 1) = x(k) + x(t)|_{t=kT_s} + \mathcal{O}(1)(T_s)
\]

where \( \mathcal{O}(1)(T_s) \) is the higher-order terms of the above expansion, i.e.

\[
\mathcal{O}(1)(T_s) = \frac{1}{2!} x^{(2)}(kT_s) T_s^2 + \cdots + \frac{1}{n!} x^{(n)}(kT_s) T_s^n + \frac{1}{(n + 1)!} x^{(n+1)}(\zeta) T_s^{n+1}, \quad \zeta \in (kT_s, kT_s + T_s)
\]

The higher-order terms of \( \mathcal{O}(1)(T_s) \) will be smaller with a higher sampling rate. Substituting (4) into (5), we have the discrete-time model

\[
x(k + 1) = x(k) + T_s f(x(k)) + T_s b(x(k)) u(k) + T_s p(x(k)) d(x(k), u(k), k) + \mathcal{O}(1)(T_s)
\]

\[
y(k) = h(x(k))
\]

A similar discretization for robust control of nonlinear plants is given in [27]. Our discrete-time estimator design will be based on the above Taylor series expansion. This paper only considers the first-order expansion, and treats the higher-order terms as disturbance. In the following analysis, the unknown input or disturbance \( d(x(k), u(k), k) \) is represented as \( d(k) \).

### 3.2. Discrete-time nonlinear state transformation

To facilitate the design of nonlinear observer, we use discrete-time nonlinear state transformation, similar to the continuous-time nonlinear state transformation [5], i.e.

\[
x(k) \rightarrow \hat{x}(k) = \hat{x}_1(k), \hat{x}_2(k), \ldots, \hat{x}_n(k)
\]

The transformation matrix is defined as

\[
\Phi = [h(x(k)) \quad L_4 h(x(k)) \ldots L_4^{(n-1)} h(x(k)) ]^T
\]

where

\[
L_4 h(x(k)) = \left[ \frac{\partial h(x)}{\partial x} \right]_{t=kT_s} = \left[ \frac{\partial h(x)}{\partial x} \right]_{t=kT_s} f(x(k))
\]

Similar to (5), using the Taylor series expansion we have

\[
\hat{x}(k + 1) - \hat{x}(k) = \hat{x}(t)|_{t=kT_s} + \mathcal{O}(2)(T_s)
\]
where $\mathcal{O}_2(T_s)$ stands for higher-order terms. It can be evaluated from (4) that

$$
\dot{\tilde{x}}(t)|_{t=kT_s} = \left[ \frac{\partial \Phi(x)}{\partial x} \right]_{x=x(kT_s)} \dot{x}(t)|_{t=kT_s} = \left[ \frac{\partial \Phi(x)}{\partial x} \right]_{x=x(kT_s)} \left[ f(x(t)) + b(x(t))u(t) + p(x(t))d(x(t), u(t), t) \right]|_{t=kT_s} = \left[ \frac{\partial \Phi(x)}{\partial x} \right]_{x=x(kT_s)} \left[ f(x(k)) + b(x(k))u(k) + p(x(k))d(k) \right]
$$

(10)

We can further obtain the following general form:

$$
\dot{\tilde{x}}(t)|_{t=kT_s} = A \tilde{x}(k) + \alpha(\tilde{x}(k)) + \gamma(\tilde{x}(k))u(k) + \tilde{p}(\tilde{x}(k))d(k)
$$

(11)

so that from the Taylor series expansion (9), we have the following transformed system:

$$
\tilde{x}(k+1) = (I + AT_s)\tilde{x}(k) + T_s\psi(\tilde{x}(k), u(k)) + T_s\tilde{p}(\tilde{x}(k))d(k) + \mathcal{O}_2(T_s)
$$

(12)

where

$$
A \triangleq \begin{bmatrix} 0 & I_{(n-1) \times (n-1)} \\ 0_{1 \times n} & 1 \end{bmatrix}
$$

is an anti-shift constant matrix, $C \triangleq [1 \ 0 \ \ldots \ 0]$ is the output matrix, $\psi(\tilde{x}(k), u(k)) \triangleq \alpha(\tilde{x}(k)) + \gamma(\tilde{x}(k))u(k)$, $\alpha(\tilde{x}(k)) \triangleq [0 \ 0 \ \ldots \ 0 \ \tilde{\zeta}(\tilde{x}(k))]^T$, $\gamma(\tilde{x}(k)) \triangleq [\gamma_1(\tilde{x}(k)) \ \gamma_2(\tilde{x}(k)) \ \ldots \ \gamma_{n-1}(\tilde{x}(k)) \ \gamma_n(\tilde{x}(k))]^T$, $\tilde{p}(\tilde{x}(k)) \triangleq [\partial \Phi(x)/\partial x]p(x(k))$, and where $\tilde{\zeta}(\tilde{x}(k)) = \tilde{L}_h(h(x(k)))$ and $\gamma_i(\tilde{x}) = L_bL_i^T h(x(k))$, for all $i = 1, \ldots, n$.

In order to establish our results, we need the following assumptions:

**Assumption 1**
The mapping $\Phi(x)$ is a diffeomorphism.

**Assumption 2**
The transferred system (12) satisfies

$$
\gamma(\tilde{x}(k)) \triangleq [\gamma_1(\tilde{x}_1(k)) \ \gamma_2(\tilde{x}_1(k), \tilde{x}_2(k)) \ \ldots \ \gamma_{n}(\tilde{x}_1(k), \tilde{x}_2(k), \ldots, \tilde{x}_n(k))]^T
$$

**Assumption 3**
The functions $\tilde{p}(\tilde{x})$, $\tilde{\zeta}(\tilde{x}(k))$ and $\gamma_i(\tilde{x}_1(k), \tilde{x}_2(k), \ldots, \tilde{x}_i(k))$, $i = 1, \ldots, n$, are global Lipschitz functions w.r.t. $\tilde{x}(k)$.

**Assumption 4**
The known functions $f(\cdot)$, $b(\cdot)$ and $p(\cdot)$ are bounded with respect to their arguments. The input of the nonlinear system (4) is bounded such that $u \leq u_{\text{max}}$ for some upper bound $u_{\text{max}}$. Further, system (4) is bounded-input-bounded-states (BIBS) stable.
Assumption 5
The disturbance input $d(\cdot)$ is observable from the output measurement, i.e. $L_p h(x) \neq 0, \forall x$.

Assumptions 2 and 3 characterize the system that is uniformly observable for any bounded input. It has been proven in [11] that Assumption 2 is a sufficient condition but not a necessary condition to ensure uniform observability for any input.

4. DISCRETE-TIME ROBUST NONLINEAR STATE ESTIMATOR

For system (4) satisfying Assumptions 1–5, a discrete-time nonlinear state estimator with the robust term incorporating current estimation can be designed as follows:

- Prediction at $t = kT_s$
  \[
  \hat{x}(k+1) = (I + AT_s)\hat{x}(k) + T_s \psi(\hat{x}(k), u(k)) + \tilde{p}(\hat{x}(k))u_v(k)
  \]  
  (13)

- Full estimation at $t = (k+1)T_s$
  \[
  \hat{x}(k+1) = \hat{x}(k+1) + L[y(k+1)-C\hat{x}(k+1)]
  \]  
  (14)

The implementation of the current estimator is similar to [22], where the output correction is based on the current sample instead of the previous sample. That is,

(i) At time $t=kT_s$, (13) predicts the partially estimated states $(\hat{x}(k+1))$ based on the previous sample’s fully estimated states $(\hat{x}(k))$.

(ii) At time $t=kT_s+T_s$, when the output $y(k+1)$ is measured, (14) gives a fully estimated $\hat{x}(k+1)$ based on $\hat{x}(k+1)$ and $y(k+1)$.

Here

\[
L = [l_1 \ l_2 \ \ldots \ l_{n-1} \ l_n]^T
\]  
(15)

is a properly chosen constant feedback estimation gain.

The design of the robust term $u_v(k)$ is based on the sliding mode theory, and is given as

\[
u_v(k) = -\frac{\rho}{L_p h(\hat{x}(k))} \text{sat}(e_1, \bar{e}) = \frac{\rho}{L_p h(\hat{x}(k))} \text{sat}(y(k) - h(\hat{x}(k)), \bar{e})
\]  
(16)

where $\tilde{p}_1(\hat{x}(k)) = L_p h(\hat{x}(k)) = [c_h(x)/\hat{x}]_{k-1} \hat{x}(k) \text{p}(\hat{x}(k))$ is the first element of the distribution vector \[\hat{p}(\hat{x}(k)) = [\hat{p}_1 \ \hat{p}_2 \ \ldots \ \hat{p}_n]^T\] in the transformed domain. According to Assumption 5, $\tilde{p}_1(\hat{x}(k)) = L_p h(\hat{x}(k)) \neq 0$. In (16), $\rho$ is a finite sliding mode estimation gain and $\bar{e}$ defines the boundary layer of the sliding manifold, both are to be given later in Section 4.2. A similar design of sliding mode gain was considered in [24, 28, 29] for continuous and discrete-time observers design.

Remark 1
The robust term (16) is similar to the general switching term employed in the literature, but is normalized w.r.t. $\tilde{p}_1(\hat{x})$, the first element of the transformed distribution vector $\hat{p}(\hat{x})$. If Assumption 5 does not hold, i.e. if $\tilde{p}_1(\hat{x}(k)) = 0$, then the unknown input is not observable, and the observer cannot overcome the unknown input/disturbance in the estimation.
In what follows, we shall prove the boundedness of the estimation error system and later establish the conditions for the existence of DSM. A relation between sliding mode gain and boundary layer thickness is also established. The estimation system rejects the disturbances/uncertainties in the sliding mode and robustness of the system is increased.

4.1. Boundedness of error dynamics

We define the estimation residual or error as

\[ e(k) = \hat{x}(k) - \hat{x}(k) = [e_1(k) \ e_2(k) \ \ldots \ e_n(k)]^T \]  

(17)

We begin with the error analysis by computing the difference of (12) and (13) as

\[ \hat{x}(k + 1) - \hat{x}(k + 1) = (I + AT_s)[\hat{x}(k) - \hat{x}(k)] + T_s\phi(\hat{x}(k), \hat{x}(k), u(k)) - C_2(T_s) \]

(18)

where \( \phi(\hat{x}(k), \hat{x}(k), u(k)) = \psi(\hat{x}(k), u(k)) - \psi(\hat{x}(k), u(k)) \).

From (14), we have

\[ \hat{x}(k + 1) - \hat{x}(k + 1) = \hat{x}(k + 1) - \hat{x}(k + 1) + L[y(k + 1) - C \hat{x}(k + 1)] \]

(19)

Substituting (18) into (19) we obtain

\[ e(k + 1) = Me(k) - N(T_s \phi(\hat{x}(k), \hat{x}(k), u(k)) - C_2(T_s)) \]

\[ + N[\hat{p}(\hat{x}(k))u_1(k) - T_s \hat{p}(\hat{x}(k))d(k)] \]

(20)

where \( M = [I + AT_s - L(C + CAT_s)] \) and \( N = [I - LC] \).

In the following Lemma 1 we analyse the boundedness of the estimation error dynamics (20) using the Lyapunov function analysis. To proceed with the analysis, we need to design \( L \) such that the eigenvalues of \( M \) are within the unit circle, then we can define a discrete-time Lyapunov function \( V(k) = e^T(k)Qe(k) \), where the constant matrix \( Q = Q^T > 0 \) is such that

\[ M^TQM - Q = -I \]  

(21)

Lemma 1

For system (4) satisfying Assumptions 1–5, estimator (13)–(14) will yield bounded \( \hat{x}(k) \) and \( e(k) \) provided that

\[ \lambda_{\text{max}}(Q) \leq \frac{1}{C_1} \]  

(22)

where \( C_1 = \|N\|^2[T_s l_\phi + l_p b_p]^2 + 2\|M\|\|N\|[T_s l_\phi + l_p b_p] \), for some Lipschitz constants \( l_\phi \), \( l_p \) and some upper bound \( b_p \).

Proof

Refer to Appendix A.1

Due to discretization and limited sampling period (\( T_s \)), the error will finally settle to a bound dictated by \( b_e \), \( T_s \) and disturbance bound \( \hat{d} \). From (A5), it is clear that the constants \( C_1 \), \( C_2 \) and
C_3 are functions of the sampling period. Provided the sampling period (T_s) is low, the error bound can be largely reduced.

4.2. Main result: existence of DSM

This paper takes two steps to improve estimation accuracy by DSM design:

(a) Define the sliding surface,
\[ e_1(k) = 0 \] (23)
and design the sliding mode estimation as in (16) to reach and remain in the sliding mode.

(b) Ensure the estimation error \( e \) goes to a bound in the sliding mode.

The following Lemma 2 and Theorem 1 are devoted to these two steps. Lemma 2 proves the convergence of \( e_1(k) \) to a bound \( e \) inside the boundary layer. Lemma 2 is the main result. It proves the existence of DSM and guarantees the confinement of trajectory within the boundary layer once it enters it.

To obtain Lemma 2, we shall first analyse the dynamics of \( e_1(k) \) of the error dynamics \( e(k) \) derived in (20). From (16), we already know the corresponding function of \( \Phi(x(k)) \) for \( e_1(k+1) \) as \( \Phi(x) = L_p h(x) \). With A, C defined in (12), and L in (15) we can compute

\[
(I - LC) = \begin{bmatrix}
1 - l_1 & 0 & \cdots & 0 \\
-l_2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-l_n & 0 & \cdots & 1 \\
\end{bmatrix}, \quad (I + AT_s) = \begin{bmatrix}
1 & T_s & 0 & \cdots & 0 \\
0 & 1 & T_s & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\] (24)

From the above, we can compute the first row of M matrix to evaluate the first dynamics \( e_1(k+1) \) from (20) as follows:

\[
e_1(k+1) = [(1-l_1)(1-l_1)T_s \ 0 \ \cdots \ 0] e(k)
+ (1-l_1)[0 + T_s(\gamma_1(\hat{x}(k)) - \gamma_1(\hat{x}(k)))u(k) - \mathcal{C}_2^R(T_s)]
+ (1-l_1)[L_p h(\hat{x}(k))u_t - T_sL_p h(x(k))d(k)]
\]

Here, \( \mathcal{C}_2^R(T_s) \) represents the higher-order terms corresponding to \( e_1(k+1) \). Re-arranging, and substituting (16), we have

\[
e_1(k+1) = (1-l_1)e_1(k) + (1-l_1)[T_s(\gamma_1(\hat{x}(k)) - \gamma_1(\hat{x}(k)))u(k)]
- (1-l_1)\rho \text{ sat}(e_1(k), \varepsilon) + (1-l_1)[T_s e_2(k) - T_sL_p h(x(k))d(k) - \mathcal{C}_2^R(T_s)] \] (25)

Lemma 2

Consider system (4) satisfying Assumptions 1–5 and estimator (13)–(14) satisfying (22). The sliding mode estimation (16) ensures that the sliding trajectory \( e_1(k) \) satisfies conditions (1) and

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(2) if outside the boundary layer (i.e. $|e_1(k)| > \varepsilon$) and that the trajectory $e_1(k)$ is finally confined within the boundary layer once it enters the boundary layer (i.e. if $|e_1(k)| < \varepsilon$, then $|e_1(k+i)| < \varepsilon$, $\forall i \in N$), provided the gain in (16) satisfies

$$\rho^- < \rho < 2\varepsilon - \rho^-$$

(26)

and $T_s l_{i1} u_{\text{max}}/(1 + T_s l_{j1} u_{\text{max}}) < l_1 < 1$, where $l_{j1}$ is the global Lipschitz constant of $\gamma_1(\cdot)$ in (13) and

$$\rho^- = \sup |\zeta(k)|, \quad \zeta(k) = T_s e_2(k) - T_s L_p h(x(k))d(k) - \mathcal{C}_2^s(T_s)$$

(27)

**Proof**

The existence of DSM outside the boundary layer and the boundedness of error dynamics inside the boundary layer are analysed separately. The analysis and the design of sliding mode gain for the existence of DSM are performed outside the boundary layer.

**Outside the boundary layer.** In our case, $s(k) = e_1(k)$. For the trajectory $e_1(k)$ outside the boundary, i.e. $|e_1(k)| > \varepsilon$, we have from (3)

$$\text{sat}(e_1(k), \varepsilon) = \text{sign}(e_1(k))$$

(28)

- **Lower bound for sliding mode gain $\rho$:** From (25) with (16) and (28), we get

$$[e_1(k+1) - e_1(k)] \text{sign}(e_1(k))$$

$$= -l_1|e_1(k)| + (1 - l_1)T_s[\gamma_1(\hat{s}(k)) - \gamma_1(\bar{s}(k))]u(k) \text{sign}(e_1(k))$$

$$+ (1 - l_1)[T_s e_2(k) - T_s L_p h(x(k))d(k) - \mathcal{C}_2^s(T_s)] \text{sign}(e_1(k)) - (1 - l_1)\rho$$

(29)

Since $x$ is bounded according to the BIBS property in Assumption 4, under the condition of Lemma 1 that ensures the boundedness of $e_2$, there exists a finite gain

$$\rho > \rho^-; \quad \rho^- = \sup |T_s e_2(k) - T_s L_p h(x(k))d(k) - \mathcal{C}_2^s(T_s)|$$

(30)

such that when $e_1(k) \neq 0$, we have

$$[e_1(k+1) - e_1(k)] \text{sign}(e_1(k)) < -l_1|e_1(k)| + (1 - l_1)T_s l_{j1} u_{\text{max}}|e_1(k)|$$

Under the globally Lipschitzian condition in Assumption 3 and the bounded input condition in Assumption 4, we have

$$[e_1(k+1) - e_1(k)] \text{sign}(e_1(k)) < -l_1|e_1(k)| + (1 - l_1)T_s l_{j1} u_{\text{max}}|e_1(k)|$$

for some Lipschitz constant $l_{i1}$ and upper bound $u_{\text{max}}$. With a proper design of the feedback estimation gain $l_1$ such that $l_1 > T_s l_{j1} u_{\text{max}}/(1 + T_s l_{j1} u_{\text{max}})$, inequality (1) for the existence of DSM can be met.

- **Upper bound for sliding mode gain $\rho$:** Similar to (29), we can show that

$$[e_1(k+1) + e_1(k)] \text{sign}(e_1(k)) = (2 - l_1)|e_1(k)| + (1 - l_1)\zeta(k) \text{sign}(e_1(k)) - (1 - l_1)\rho$$

$$+ (1 - l_1)T_s[\gamma_1(\hat{s}(k)) - \gamma_1(\bar{s}(k))]u(k) \text{sign}(e_1(k))$$

In order for the above equation to meet condition (2), we need to have

$$\rho < \min \left\{ \frac{2 - l_1}{1 - l_1} |e_1(k)| + T_s[\gamma_1(\hat{s}(k)) - \gamma_1(\bar{s}(k))]u(k) \text{sign}(e_1(k)) + \zeta(k) \text{sign}(e_1(k)) \right\}$$

(31)
From the above analysis we already have
\[ |\gamma_1(\hat{x}(k)) - \gamma_1(x(k))|u(k)| \leq T_s l_{g_1} u_{\max} |e_1(k)| \leq \frac{l_1}{1-l_1} |e_1(k)| \]
\[ |\zeta(k)\text{ sign}(e_1(k))| \leq \rho^- \]
where \( \rho^- = \sup |\zeta(k)| \). The right-hand side of (31) satisfies:
\[ \frac{2 - l_1}{1 - l_1} |e_1(k)| + T_s [\gamma_1(\hat{x}(k)) - \gamma_1(x(k))] u(k) \text{ sign}(e_1(k)) + \zeta(k) \text{ sign}(e_1(k)) \]
\[ \geq \frac{2 - l_1}{1 - l_1} |e_1(k)| - \frac{l_1}{1 - l_1} |e_1(k)| - \rho^- \]
\[ = 2|e_1(k)| - \rho^- \]
For the trajectory outside the boundary layer, i.e. \( |e_1(k)| > \varepsilon \), the upper bound for the sliding mode gain \( \rho^+ \) can be derived as
\[ \rho^+ = 2\varepsilon - \rho^- \] (32)
So finally the gain should be designed as \( \rho^- < \rho < \rho^+ \) to meet both conditions (1) and (2). The above conditions ensure the convergence of \( e_1(k) \) until \( e_1(k) \) reaches the boundary layer.

The selection of \( \rho \) forms the basis for preventing switching across the sliding manifold. Once the trajectory enters the boundary layer, the switching is changed to feedback correction and the trajectory stays inside the boundary layer forever. This phenomenon is illustrated in Figure 1.

Inside the boundary layer. Inside the boundary layer, we have \( |e_1(k)| < \varepsilon \) and \( \text{sat}(\cdot, \varepsilon) = (\cdot)/\varepsilon \). The error dynamics (25) can be reformulated as
\[ e_1(k + 1) = (1 - l_1)e_1(k) + (1 - l_1)\zeta(k) - (1 - l_1)\rho \frac{e_1(k)}{\varepsilon} \]
\[ + (1 - l_1)T_s [\gamma_1(\hat{x}(k)) - \gamma_1(x(k))] u(k) \]
Since \( \rho^- > \sup |\zeta(k)| \) and \( l_1 > (1 - l_1)T_s l_{g_1} u_{\max} \), the following can be shown easily:
\[ |e_1(k + 1)| < \left[ (1 - l_1) \frac{\rho}{\varepsilon} - 1 \right] |e_1(k)| + l_1 |e_1(k)| + (1 - l_1)\rho^- \]
\[ < (1 - l_1)(\rho - \varepsilon) + l_1\varepsilon + (1 - l_1)\rho^- \]
From condition (26), we need to have \( \varepsilon > \rho^- \), and from (32), \( \rho^+ - \varepsilon = \varepsilon - \rho^- \). Thus, \( |\rho - \varepsilon| < \varepsilon - \rho^- \), and it can be readily shown that
\[ |e_1(k + 1)| < (1 - l_1)(\varepsilon - \rho^-) + l_1\varepsilon + (1 - l_1)\rho^- \]
\[ = \tilde{\varepsilon} \]
Hence, condition (26) confines the error \( e_1 \) to an ultimate bound inside the boundary layer. \( \square \)

In the ideal case of \( T_s = 0, b_{\varepsilon} = 0 \) and the system will remain on the sliding surface. But due to limited sampling rate \( (T_s^{-1}) \) the error will finally remain in a small bound that is determined by the disturbance \( d(k) \), the higher-order dynamics \( \dot{c}_2(T_s) \) and the sampling period \( T_s \).

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Remark 2
From the definition of $\rho^-$ in (27) we note that $\rho^-$ is a (polynomial) function of $T_s$, hence $\varepsilon$ can be made sufficiently small when sampling rate $1/T_s$ is sufficiently high.

4.3. Convergence of error dynamics in the sliding mode

We need to check the convergence of other state variables. Since the robust term (16) ensures the existence of DSM, we only need to check the convergence of $e$ in the sliding mode.

In the ideal sliding mode, $e_1(k+1) = e_1(k) = 0$ and $\hat{x}_1(k) = \bar{x}_1(k)$, the discrete equivalent control $u^e_1(k)$ can be obtained from (25) and (16) as

$$0 = (1 - l_1)T_s \times e_{2d}(k) - (1 - l_1)C^e_2(T_s) + (1 - l_1)[L_p h(\hat{x}_d(k))u^e_1(k) - T_s L_p h(x(k))d(x(k))]$$

Hence

$$u^e_1(k) = \frac{C^e_2(T_s) - T_s \times e_{2d}(k)}{L_p h(\hat{x}_d(k))} - \frac{T_s L_p h(x(k))d(k)}{L_p h(\hat{x}_d(k))} (33)$$

where the subscript $d$ denotes the estimated $\hat{x}$-related variables in the sliding mode, i.e. $e_d(k) = [e_{1,d}(k) \ e_{2,d}(k) \ldots e_{n,d}(k)]^T = \hat{x}_d(k) - \bar{x}(k)$ and $\hat{x}_d = \Phi^{-1}(\bar{x}_d(k))$.

Remark 3
From (33), it can be clearly seen that, the equivalent control of the robust term is directly related to the unknown input/disturbance. Once the system reaches the sliding mode, the unknown input can be rejected in the state estimation. The unknown input can also be estimated from the above equivalent control (33), and its estimation will be discussed in the next subsection.

Substituting (33) into (20), we obtain the estimation error dynamics in the sliding mode of $e_1 = 0$, i.e.

$$e_d(k+1) = M e_d(k) - N[T_s \phi(\bar{x}(k), \hat{x}_d(k), u(k)) - C_2(T_s)]$$

$$+ N[\bar{p}(\hat{x}_d(k)) - \bar{p}(x(k))]u^e_1(k) + N[\bar{p}(x(k))u^e_1(k) - T_s d(k)] (34)$$

Obviously, the equilibrium point of the error dynamics (34) is $e_d(k) = 0$, i.e. $\hat{x}_d(k) = x(k)$. The following theorem defines the condition for asymptotic stability of the estimation error.

Theorem 1
Consider system (4) satisfying Assumptions 1–5 and estimator (13)–(14) with sliding mode estimation (16) and gain (26). The estimation error is globally asymptotically stable in the sliding mode $e_1(k+1) = e_1(k) = 0$ provided the gain $L$ satisfies $T_s l_1 u_{\max}/(1 + T_s l_1 u_{\max}) < l_1 < 1$, and

$$\lambda_{\max}(Q) < \min \{C_1^{-1}, 2||N||||M||C_b + ||N||^2 C_b^2)^{-1}\} (35)$$

where $C_1$ is defined in Lemma 1 and $C_b = T_s l_1 l_\phi + l_\phi \bar{u}^e_1 + T_s b_1 b_2 b_1 / \bar{p}_1 (1 + l_\phi d)$ for Lipschitz constants $l_\phi$, $l_\phi$, $l_\phi$, and upper bounds $\bar{u}^e_1$, $\bar{d}$, $b_2$, $b_1 / \bar{p}_1$.

Proof
See Appendix A.2. \(\square\)
From inequality (A11), if $\mathcal{L}_2(T_s) = 0$, then $h_\ell = 0$, and condition (35) ensures global asymptotic stability of $e(k)$. Due to limited sampling rate, $\mathcal{L}_2(T_s) \neq 0$, condition (35) can only ensure the global ultimate boundedness of $e(k)$. A small $\mathcal{L}_2(T_s)$ will yield a smaller ultimate bound. It is also clear from the proof of Theorem 1 that the effect of unknown input is cancelled in the state estimation by the presence of robust term.

**Remark 4**
The positive constants $C_1$ in Lemma 1 and $C_b$ in Theorem 1 are dependent on the sampling period $T_s$. In fact, they can be rewritten as $C_1 = T_s \times f_{c_1}()$ and $C_b = T_s \times f_{c_b}()$ where $f_{c_1}()$ and $f_{c_b}()$ are polynomial functions of $T_s$. Hence, the selection of a lower sampling period $T_s$ decreases the bounds in inequalities (22) and (35).

**4.4. Unknown input estimation from sliding mode**

Once the trajectories reaches the sliding mode and all the states converges to the true states, the equivalent control $u_{eq}$ information can be used to reconstruct the unknown input to a certain level of accuracy. Once the system attains DSM, and states converge to the true states, we have $\hat{x}_d(k) \rightarrow \hat{x}(k)$. Therefore

$$e_{2,d} \approx 0, \quad L_p h(\hat{x}_d(k)) \approx L_p h(\hat{x}(k))$$

Therefore, from (33) we can approximate

$$u^*_r(k) \approx \frac{\mathcal{L}_2^*(T_s)}{L_p h(\hat{x}_d(k))} + T_s d(k) \quad (36)$$

Although ideal sliding mode cannot be attained in DSM, we are still able to approximate the unknown input. By neglecting the higher-order terms, we have

$$\dot{\hat{d}}(k) \approx \frac{(u^*_r(k))_{eq}}{T_s} \quad (37)$$

We can recover the equivalent control signal by the use of a low-pass filter [30]. Continuous approximation of equivalent injection signal by using a small positive scalar was implemented in [31]. Alternatively, by using a small boundary layer thickness $\epsilon$, we can eliminate chattering and also approximate the discontinuous component of the signal within the boundary layer. The usage of boundary layer is also equivalent to the approximation given by Edwards *et al.* [31]. Similar to the approximation of [31], for some small positive scalar $\delta$, the unknown input can be estimated as

$$\dot{\hat{d}}(k) \approx \frac{1}{T_s} \left( \frac{\rho}{L_p h(\hat{x}(k))} \text{sat}(e_1, \epsilon) \right)_{eq}$$

$$\approx \frac{\rho}{T_s L_p h(\hat{x}(k))} \frac{e_1}{(|e_1| + \delta)} \quad (38)$$

**Remark 5**
The estimation of unknown input relies only on the output estimation error and hence the estimation can be performed online together with the state estimation.
4.5. Selection of sampling period $T_s$ and boundary layer thickness $\varepsilon$

The selection of sampling period in the proposed observer design plays a crucial role in the accuracy of state and unknown input estimations. The proposed DSM determines the sliding mode gain $\rho$ based on the boundary layer thickness $\varepsilon$, which in turn is related to the magnitude of the disturbance and the sampling period $T_s$.

For a given sampling period, a boundary layer can be selected according to (22), such that within a sampling period, the trajectory cannot cross the boundary layer despite the presence of unknown input. It also reveals that the estimation accuracy is dependent on the frequency of the unknown input. In order to have good accuracy in the proposed DSM, the sampling frequency $(1/T_s)$ should be higher than the highest frequency component of the disturbance/unknown input. This ensures the confinement of the error trajectory within a small boundary layer compared to unknown input dynamics.

If the sampling frequency of the system is small compared to the highest frequency components of unknown input, then a large boundary layer will be required, and the fast dynamics of the unknown inputs may not be accurately estimated. The accuracy of the state estimation will be poor. The sampling period should be chosen appropriately such that the robust term is able to track the unknown input accurately. The accuracy of the unknown input estimation also relies on the sampling period $T_s$ as the higher-order dynamics $C^s_z(T_s)$ is dependent on $T_s$ (36).

4.6. Discrete-time state estimator in the original space

In the previous sections, we have presented the design of the nonlinear observer in the transformed domain so that the analysis can be readily carried out. However, our purpose is to produce the estimation of the original system (4). Hence we need to transfer (13) from the transformed domain back to the original space. As the transformation matrix $\Phi(x(k))$ is a diffeomorphism according to Assumption 1, the Jacobian matrix $\frac{\partial \Phi(x)}{\partial x}$ and the inverse Jacobian matrix $[\frac{\partial \Phi(x)}{\partial x}]^{-1}$ exist. From (10) and (11) we can have

$$\left[\frac{\partial \Phi(x)}{\partial x}\right]^{-1}_{t=kT_s} (AT_s\hat{x}(k) + \psi(\hat{x}(k)) + \gamma(\hat{x}(k))u(k)) = T_s f(x(k)) + T_s b(x(k))u(k)$$

(39)

For the current estimator described by (13) and (14) in the combined form, we have from (39)

$$\hat{x}(k + 1) - \hat{x}(k) = \left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\hat{x}(k)}^{-1} [T_s f(\hat{x}(k)) + T_s b(\hat{x}(k))u(k) + p(\hat{x}(k))u_t(k)]$$

$$+ L[y(k + 1) - h(\hat{x}(k + 1))]$$

(40)

Since $[\frac{\partial \Phi(x)}{\partial x}]_{x=\hat{x}(k)}^{-1} [\hat{x}(k + 1) - \hat{x}(k)] = \hat{x}(k + 1) - \hat{x}(k)$, from (40) we have the following estimator in the original space:

$$\hat{x}(k + 1) - \hat{x}(k) = T_s f(\hat{x}(k)) + T_s b(\hat{x}(k))u(k)$$

$$+ p(\hat{x}(k))u_t(k) + \left[\frac{\partial \Phi(x)}{\partial x}\right]_{x=\hat{x}(k)}^{-1} L[y(k + 1) - h(\hat{x}(k + 1))]$$

(41)

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The estimator in the original space can therefore be achieved in two steps:

1. **Prediction of partial states at instant ‘k’**:

\[
\dot{x}(k+1) = \dot{x}(k) + T_s f(\dot{x}(k)) + T_s b(\dot{x}(k))u(k) \\
+ p(\dot{x}(k)) \frac{\rho}{L_p h(\dot{x}(k))} \text{sat}[y(k) - h(\dot{x}(k)), \varepsilon]
\]  

(42)

2. **Full estimation incorporating current estimator at instant ‘k + 1’**:

\[
\dot{x}(k+1) = \ddot{x}(k+1) + \left[ \frac{\partial \Phi(x)}{\partial x} \right]_{x=\ddot{x}(k)}^{-1} \mathbf{L}[y(k+1) - h(\ddot{x}(k+1))]
\]  

(43)

**Remark 6**

In practice, the observer design can be directly performed in the original space. The system is transformed into the observation space for ease of analysis.

4.7. **Design of gain L**

The gain \( \mathbf{L} \) should be designed to meet the Lyapunov condition (21) and the two conditions in (35). The stability aspects for observers of the similar kind are discussed in [32]. The methods of [8, 33, 34] provide some insights into the distance to unobservability and the selection of the condition number that gives better stability. In our design, we need to calculate the Lipschitz constants and bounds for the functions in the transformed space. Then, we need to design the observer such that the distance to unobservability of the linear matrix \( \mathbf{I} + AT_s - \mathbf{LC} - \mathbf{LCAT}_s \) is greater than the Lipschitz constants and the bounded functions in (35) in order to achieve stability for the system.

As an alternative approach, we may use the LMI approach [35] to solve the gain design problem. The iterative procedure is as follows:

1. We first design gain \( \mathbf{L} \) such that \( 1 > l_1 > T_s l_1 u_{\text{max}}/(1 + T_s l_1 u_{\text{max}}) \) and the eigenvalues of \( \mathbf{M} \) are within the unit circle.
2. The Lyapunov condition (21) can be rewritten in the form of matrix inequality in variable \( \mathbf{Q} \) as follows:

\[
\mathbf{M}^T \mathbf{Q} \mathbf{M} - \mathbf{Q} \leq 0
\]  

(44)

\[
\mathbf{Q} > 0
\]  

(45)

The matrix \( \mathbf{Q} \) should be designed such that it satisfies the inequalities in (35).
3. Once the gain \( \mathbf{L} \) and Lipschitz constants are known, the inequalities in (35) can be evaluated. The inequalities in (35) can be formed in variable \( \mathbf{Q} \) as

\[
\mathbf{Q} < \mathbf{I} \times \min \left\{ \frac{1}{C_1}, \frac{1}{K_2} \right\}
\]  

(46)

where \( K_2 = \{2||N||||\mathbf{M}||C_b + ||N||^2C_b^2\} \).
4. From the above, we can express the linear matrix inequality in the variable $Q$ as

$$Q > 0, \begin{bmatrix} Q & M^TQ \\ QM & Q \end{bmatrix} > 0, \quad Q < I \times \min \left\{ \frac{1}{C_1}, \frac{1}{K_2} \right\}$$

(47)

5. If there does not exist a $Q$ satisfying the above LMI problem, the sampling rate will be increased and the above steps are repeated.

5. APPLICATION TO STATE ESTIMATION OF A BIOREACTOR

As an example, we study the state estimation in a bioreactor, which has been discussed in many works [11, 12, 36, 37]

$$\dot{X} = \frac{\mu_m X S}{k_c X + S} - XD$$

(48)

$$\dot{S} = -\frac{1}{Y} \frac{\mu_m X S}{k_c X + S} + (S_f - S)D$$

The system satisfies: (1) the dilution rate is bounded as $D(t) \geq D_{\text{min}} > 0$, $\forall t$, where $D_{\text{min}}$ is a constant; (2) the feed rate $S_f$ is bounded; (3) each reaction involves at least one reactant which is neither a catalyst nor an autocatalyst, hence according to Theorem 1.1 in [36], we can conclude that the state variables $X(t) > 0$ and $S(t) > 0$ are bounded for all $t$. Hence Assumption 4 is satisfied.

The cell mass concentration $X(t)$ is assumed to be measurable. The target is to estimate the unavailable $S(t)$ from the measurable $X(t)$. The growth function in general is uncertain. In practice, $\mu_m$ and $k_c$ are uncertain and time varying. Hence we can model the uncertainties as

$$\mu_m = \mu_m^0 + d_1(t), \quad k_c = k_c^0 + d_2(t)$$

where $\mu_m^0$ and $k_c^0$ are known nominal parameters, and $d_1(t)$ and $d_2(t)$ model the bounded additive time-varying parametric uncertainties. So, the uncertainties can be factored out easily.

We can express (48) in the general form of (4), i.e.

$$x = \begin{bmatrix} X \\ S \end{bmatrix}, \quad f(\cdot) = \begin{bmatrix} \frac{\mu_m^0 S}{k_c^0 X + S} \\ -\frac{1}{Y} \frac{\mu_m^0 S}{k_c^0 X + S} \end{bmatrix}, \quad b(\cdot) = \begin{bmatrix} -X \\ S_f - S \end{bmatrix}, \quad p(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d(x) = \frac{\mu_m S}{k_c X + S} - \frac{\mu_m^0 S}{k_c^0 X + S} \quad u = D, \quad y = h(x) = X, \quad \frac{\partial h(x)}{\partial x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L_p h(x) = 1$$

The state transformation $\dot{x} = \Phi(x) = [h(x) \quad L_f h(x)]^T$ produces

$$\Phi(x) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} X \\ \frac{\mu_m^0 X S}{k_c^0 X + S} \end{bmatrix}$$
Hence, we have
\[
\frac{\partial \Phi(x)}{\partial X} = \begin{bmatrix}
1 & 0 \\
\mu_m^0 S^2 & \mu_m^0 \frac{X^2}{Y^2} \\
(\mu_m^0 X + Y)^2 & (\mu_m^0 X + Y)^2
\end{bmatrix}, \quad \frac{\partial \Phi(x)}{\partial X}^{-1} = \begin{bmatrix}
1 & 0 \\
-\frac{S^2}{K_c^0 X^2} & \left(\mu_m^0 X + Y\right)^2
\end{bmatrix}
\]

Therefore, Assumption 1 is satisfied. The discrete-time system in the observation space can be obtained as
\[
\begin{bmatrix}
\hat{X}(k+1) \\
\hat{S}(k+1)
\end{bmatrix} = \begin{bmatrix}
\hat{X}(k) \\
\hat{S}(k)
\end{bmatrix} + \begin{bmatrix}
0 & T_s \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\hat{X}(k) \\
\hat{S}(k)
\end{bmatrix} + \begin{bmatrix}
0 \\
\xi(\hat{X}(k), \hat{S}(k))
\end{bmatrix}
 + \begin{bmatrix}
-T_s \hat{X}(k) \\
\gamma(\hat{X}(k), \hat{S}(k))
\end{bmatrix} D(k) + c_2(T_s) + T_s \left[\frac{\partial \Phi(x)}{\partial X}\right] \begin{bmatrix}
1 \\
-\frac{1}{Y}
\end{bmatrix} d(k)
\]

It can be verified that \(\xi(\hat{X}, \hat{S}), \gamma(\hat{X}, \hat{S})\) are differentiable with respect to their arguments, this together with the boundedness of the states \(X, S, \hat{X}, \hat{S}\) ensures \(\xi\) and \(\gamma\) are global Lipschitz functions w.r.t. \(x\). Moreover, the above equation belongs to the structure given in (12). Hence both Assumptions 2 and 3 are satisfied. As the functions \(p = [1 \ -1/Y]^T\) and \(L_p h(x) = 1\) are constant, they are global Lipschitz functions. Hence Assumption 5 is also satisfied.

Based on the design of feedback gain in Section 4.7, we can select a proper estimation gain as \(L = [l_1 \ l_2]^T\). The sliding mode estimation gain \(\rho\) and boundary layer thickness \(\varepsilon\) are selected according to condition (26). The proposed discrete time robust estimator is designed in accordance with (42) and (43) as

1. **Prediction phase:**
\[
\hat{X}(k+1) = \hat{X}(k) + \frac{T_s \mu_m^0 \hat{X}(k) \hat{S}(k)}{K_c^0 \hat{X}(k) + \hat{S}(k)} - T_s \hat{X}(k) D(k) + \rho \operatorname{sat}[y(k) - h(\hat{S}(k)), \varepsilon]
\]

2. **Full estimation phase:**
\[
\begin{align*}
\hat{X}(k+1) &= \hat{X}(k+1) + l_1 (y(k+1) - \hat{X}(k+1)) \\
\hat{S}(k+1) &= \hat{S}(k+1) + \left[\frac{(\mu_m^0 \hat{X}(k) + \hat{S}(k))^2}{\mu_m^0 \frac{K_c^0}{Y} \hat{X}(k)} - \frac{\hat{S}(k)^2}{K_c^0 X^2(k)} l_2 \right] (y(k+1) - \hat{X}(k+1))
\end{align*}
\]
The unknown input or disturbances can be estimated according to (38) as

$$\hat{d}(x(k)) \approx \frac{1}{T_s} \frac{(\dot{X}(k) - X(k))}{C_0 X(k)} + \delta$$

(53)

5.1. Simulation results and discussions

The parameter values chosen for the numerical simulation are tabulated in the Table I.

The sampling period $T_s$ for simulation is chosen to be 0.05 h. The additive parametric uncertainties $d_1(t)$ and $d_2(t)$ are chosen to be $0.4 \sin(\pi t)$ and $0.3 \sin(3\pi t)$ respectively. The uncertainties chosen are relatively high as compared to their nominal values, and disturbance $d(k)$ is shown in Figure 2. The control input, i.e. the dilution rate is chosen as

$$D(t) = \begin{cases} 
0.2 & 0 < t \leq 10 \\
0.6 & 10 < t \leq 20 \\
0.2 & 20 < t \leq 30 
\end{cases}$$

(54)

Based on the bounds of disturbances and states available, the Lipschitz constants and bounds of functions are evaluated. We obtain $l_{u_1} = 0.05$, $u_{\text{max}} = 0.6$, $|e_2(k)| \leq 0.04$ and $|d(x)| \leq 0.3$. So, we have $T_s l_{u_1} u_{\text{max}} / (1 + T_s l_{u_1} u_{\text{max}}) = 0.003$, $\rho^* = 0.016$. Hence according to Lemma 2, we choose

Table I. Initial conditions and model parameter values chosen for simulation of bioreactor.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{\text{m}}^0$</td>
<td>1 h$^{-1}$</td>
</tr>
<tr>
<td>$Y$</td>
<td>1</td>
</tr>
<tr>
<td>$X(t_0)$</td>
<td>0.05 g l$^{-1}$</td>
</tr>
<tr>
<td>$S(t_0)$</td>
<td>0.08 g l$^{-1}$</td>
</tr>
<tr>
<td>$\dot{X}(t_0)$</td>
<td>0.08 g l$^{-1}$</td>
</tr>
<tr>
<td>$S_{\text{f}}$</td>
<td>0.1 g l$^{-1}$</td>
</tr>
</tbody>
</table>

![Figure 2. Disturbance for the bioreactor example.](image-url)
\( l_1 = 0.12, \rho = 0.032 \) and \( \varepsilon = 0.025 \) to satisfy (26) in order to ensure the existence DSM. We select \( l_2 = 0.1 \). Both eigenvalues of \( I + A T_s - LC - LCA T_s \) are within the unit circle. The maximum eigenvalue of \( Q \) from the solution of (21) is 25. Hence, \( C_1 \) and \( \{2||N||M||C_b + ||N||^2C_b^2\} \) must be less than 0.04 to satisfy (35) and (22) for convergence. We evaluated the bounds and Lipschitz constants used in the analysis of Lemma 1 and Theorem 1 and verified that they satisfy the conditions (22) and (35).

We first applied the discrete-time nonlinear estimator without the robust terms, i.e. with \( \rho = 0 \) in (49)–(52). The performance is shown in Figure 3. Using the DSM approach, where the robust estimator is described by (49)–(52), there is a significant improvement in the tracking performance, as shown in Figure 4.

The proposed robust nonlinear estimator uses \( e_1 = X(k) - \dot{X}(k) = 0 \) as the sliding surface. It demonstrates that the sliding surface is reached in finite time and the trajectory is maintained inside the boundary layer thereafter. From Figure 5, it is evident that the errors \( e_1 \) and \( e_2 \) converge to the boundary layer in a finite time and stays in the vicinity of the sliding plane inside the boundary layer.

To estimate the unknown input according to (53), \( \delta \) is chosen to be 0.001. The unknown input reconstructed from the sliding mode is shown in Figure 2. The estimated unknown input converges to the actual disturbance after all the states have converged. The accuracy of the

![Figure 3. The actual and estimated cell mass/substrate concentration by discrete-time nonlinear estimator.](image-url)
Figure 4. The actual and estimated cell mass/substrate concentration by robust discrete-time nonlinear estimator.

Figure 5. Phase trajectory of errors when $\rho < 2\varepsilon - \rho^-$ with $\rho = 0.032$ and $\varepsilon = 0.025$.
unknown input estimation relies on the bound of higher-order dynamics and the sampling period $T_s$.

Since the ultimate boundedness of the error depends on the sampling period, the selection of sampling period should be appropriate to maintain accuracy in estimation, such that the robust term can track the unknown input/disturbance accurately. When sampling period is changed to $T_s = 0.1$, the error bound increases slightly as shown in Figure 6. For $T_s = 0.4$, the ultimate bound increases and estimation accuracy is decreased as the robust term cannot track the unknown input as depicted in Figure 7.

![Figure 6. Phase trajectory of errors when $T_s = 0.1$.](image1)

![Figure 7. Phase trajectory of errors when $T_s = 0.4$.](image2)
5.1.1. Discrete-time sliding mode with boundary layer. With $\rho^- = 0.016$, $\rho = 0.032$ and $\epsilon = 0.025$, condition (26), $\rho < 2\epsilon - \rho^-$ is satisfied. Chattering is completely eliminated and trajectory is confined inside the boundary layer as shown in Figure 5. If $\rho > 2\epsilon - \rho^-$, condition (26) is violated, chattering occurs across the sliding manifold. With $\rho = 0.042$ and $\epsilon = 0.025$, chattering starts to occur but the trajectory is finally confined to the boundary layer as shown in Figure 8. When $\rho = 0.051$ and $\epsilon = 0.025$, more severe chattering occurs as depicted in Figure 9. When $\rho = 0.053$ and $\epsilon = 0.025$, the chattering does not converge as shown in Figure 10. If condition (26) does not hold, the system cannot eliminate chattering and DSM does not exist. A slight variation in the sliding mode gain can alter the system stability. Hence, condition (26) plays a crucial role in the design of DSM.

![Figure 8](image1.png)  
Figure 8. Phase trajectory of errors when $\rho > 2\epsilon - \rho^-$ with $\rho = 0.045$ and $\epsilon = 0.025$.

![Figure 9](image2.png)  
Figure 9. Phase trajectory of errors when $\rho > 2\epsilon$ with $\rho = 0.051$ and $\epsilon = 0.025$.  

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6. CONCLUSIONS

In this paper, we consider the design of a DSM observer for a class of nonlinear systems to robustly estimate the system states when uncertainties or unknown input is present in the system. With our DSM design, the uncertainties and disturbances can be readily handled and a good estimation accuracy can be obtained. The unknown input or disturbances can be estimated through sliding mode. A convergence analysis has been performed in the sliding mode and necessary conditions for the existence of sliding mode are obtained. A relation between sliding mode gain and boundary layer thickness is also established for the existence of DSM. Our analysis proves that the estimation error converges under estimation feedback gain and sliding mode gain. An application to a biochemical process is given and the simulation results demonstrate the desired convergence properties.

APPENDIX A

A.1. Proof of Lemma 1

For the system satisfying Assumption 4, \(|u(k)| \leq u_{max}\), the system states \(\hat{x}(k)\) are all bounded, and \(\|C_2(T_s)\| \leq b_e\) for some upper bounds \(u_{max}\) and \(b_e\). Since \(L_{ph}(x(k)) \neq 0\), \(\forall(x)\), then \(|u_i(k)| \leq \rho/\|L_{ph}(x(k))\| \leq b_{u_i}\) for some upper bound \(b_{u_i}\). Under Assumption 3, \(\psi(\hat{x}(k), u(k))\) and \(\bar{p}(\hat{x}(k))\) are globally Lipschitzian, so we have

\[
||\psi(\hat{x}(k), u(k)) - \psi(\hat{x}(k), u(k))|| \leq l_p||e(k)|| \tag{A1}
\]

\[
||(\bar{p}(\hat{x}(k)) - \bar{p}(\hat{x}(k)))u_i(k)|| \leq l_{pu}b_{e_i}||e(k)|| \tag{A2}
\]
for are Lipschitz constants $l_\Phi$ and $l_p$. Under Assumptions 1 and 4, $\hat{p}(\hat{x}(k))$ is bounded due to the existence of $\partial \hat{p}(\hat{x})/\partial \hat{x}$ and the boundedness of $x(k)$. So, we have
\[
\|\hat{p}(\hat{x}(k))[T_s d(k) - u_t(k)]\| \leq b_p (T_s \bar{d} + b_p)
\] (A3)
for some upper bound $b_p$. Using the above results (A1)–(A3), we have
\[
\begin{aligned}
&\|\hat{p}(\hat{x}(k)) u_t(k) - T_s \hat{p}(x(k)) d(k)\|
\leq &\|\hat{p}(\hat{x}(k)) u_t(k) - \hat{p}(x(k)) u_t(k)\| + \|\hat{p}(x(k)) u_t(k) - T_s \hat{p}(x(k)) d(k)\|
\leq &\ l_p \|e(k)\| + b_p (T_s \bar{d} + b_p)
\end{aligned}
\] (A4)

Using the Lyapunov function $V(k) = e^T(k) Q e(k)$, and the obtained results (A1)–(A4), together with (21), we may evaluate the difference of Lyapunov function as
\[
V(k+1) - V(k) = e^T(k+1) Q e(k+1) - e^T(k) Q e(k)
\leq [e^T(k) [M^T P M] e(k) - e^T(k) Q e(k)]
+ \lambda_{\text{max}}(Q) C_1 \|e(k)\|^2 + \lambda_{\text{max}}(Q) C_2 \|e(k)\| + \lambda_{\text{max}}(Q) C_3
\leq - [1 - \lambda_{\text{max}}(Q) C_1] \|e(k)\|^2 + \lambda_{\text{max}}(Q) C_2 \|e(k)\| + \lambda_{\text{max}}(Q) C_3
\] (A5)

for some positive constants $C_2$ and $C_3$ defined by
\[
C_2 = \|M||N||b \epsilon + T_s \|N\|^2 b \epsilon \ l_\Phi + \|N\|^2 l_p b_p + \|M\||N||b_p (T_s \bar{d} + b_p)
+ T_s \|N\|^2 l_p b_p (T_s \bar{d} + b_p) + \|N\|^2 l_p b_p b_p (T_s \bar{d} + b_p)
\]

\[
C_3 = \|N\|^2 [b_\epsilon (b_p (T_s \bar{d} + b_p))^2 + 2b_\epsilon b_p (T_s \bar{d} + b_p)]
\]

Feedback gain is designed such that $\lambda_{\text{max}}(Q) < 1/C_1$, i.e. condition (22) is satisfied. Hence the quadratic term is negative and so $e(k)$ is bounded. As $x(k)$ is bounded, $\hat{x}(k)$ is also bounded.

### A.2. Proof of Theorem 1

Since $e_2(k)$, $x(k)$ and $d(k)$ are bounded, then according to (33), $|u_t^*| \leq \tilde{u}_t^*$ for some upper bound $\tilde{u}_t^*$. Under Assumption 3, similar to (A1) and (A2), we have
\[
\|\Phi(\hat{x}(k), \hat{x}_d(k), u(k))\| \leq l_\Phi \|e_d(k)\|
\] (A6)
\[
\|\hat{p}(\hat{x}_d(k)) - \hat{p}(x(k)) u_t^*(k)\| \leq l_p \tilde{u}_t^* \|e_d(k)\|
\] (A7)

Similar to (A3), $\|\hat{p}(x(k))\| \leq b_p$ and $1/L_p h(x(k)) \leq b_1 / \tilde{p}_1$. According to Assumption 3, $\tilde{p}_1(\hat{x}(k)) = L_p h(\hat{x}(k))$ is globally Lipschitzian. Therefore,
\[
\|L_p h(\hat{x}_d(k)) - L_p h(x(k))\| = \|\tilde{p}_1(\hat{x}_d(k)) - \tilde{p}_1(\hat{x}(k))\| \leq l_{\tilde{p}_1} \|e_d(k)\|
\] (A8)
From (33) with (A8), we may evaluate
\[
||u^e(k) - Tsd(k)|| \leq \left| \left| C_2(T_s) \left( \frac{L_p T_s e_2(k)}{L_p h(\hat{s}(k))} - T_s e_2(k) \right) \right| + \left| \left| L_p h(x(k)) \left( \frac{T_s e_2(k) + T_s d(k)}{L_p h(\hat{s}(k))} - 1 \right) \right| \right|
\leq b_c b_{1/p} + T_s b_{1/p} ||e_d(k)|| + T_s b_{1/p} l_p d ||e_d(k)||
= b_{1/p} b_c + T_s b_{1/p} (1 + l_p d ||e_d(k)||)
\]
(A9)

The error dynamics (34) could be further simplified using the above results of (A7)–(A9) to obtain
\[
||e_d(k + 1)|| 
\leq ||Me_d(k)|| + ||N||T_s \phi(\hat{s}(k), \hat{x}(k), u(k)) + C_2(T_s)||
+ ||N|| \times ||\hat{p}(x(k)) - \hat{p}(x(k))|| \times ||u^e(k)|| + ||N|| \times ||\hat{p}(x(k))|| \times ||u^e(k) - T_s d(k)||
\leq ||M|| ||e_d(k)|| + T_s b_0 ||N|| ||e_d(k)|| + ||N|| ||b_c + l_p r^e_1|| ||e_d(k)||
+ ||N|| ||b_c b_{1/p} + T_s b_{1/p} (1 + l_p d ||e_d(k)||) ||e_d(k)||
= ||M|| ||e_d(k)|| + ||N|| ||e_d(k)|| ||C_b + ||N|| ||b_c C_l||
(A10)

where \(C_b = T_s b_0 + l_p r^e_1 + T_s b_{1/p} (1 + l_p d)\), \(C_l = b_c b_{1/p} + 1\).

Now, we choose Lyapunov function candidate as \(V_3(k) = e_d^T(k)Qe_d(k)\), and from (34) together with (A10), we have the difference of Lyapunov function as
\[
V_3(k + 1) - V_3(k) 
= e_d^T(k + 1)Qe_d(k + 1) - e_d^T(k)Qe_d(k)
\leq [e_d^T(k) [M^T Q M] e_d(k) - e_d^T(k)Qe_d(k)] + 2 \lambda_{\text{max}}(Q) ||N|| ||M|| ||C_b|| ||e_d(k)||^2
+ 2 \lambda_{\text{max}}(Q) ||N|| ||b_c C_l + ||N|| ||C_b C_l|| ||e_d(k)||
+ \lambda_{\text{max}}(Q) ||N|| ||b^2 C_l^2 + \lambda_{\text{max}}(Q) ||N|| ||C_b^2 || ||e_d(k)||^2
\leq -[1 - \lambda_{\text{max}}(Q) (2 ||N|| ||M|| ||C_b + ||N|| ||C_b^2|| ||e_d(k)||^2
+ \lambda_{\text{max}}(Q) ||b_c || (2 ||N|| ||C_b + ||N|| ||C_b^2|| ||e_d(k)|| + \lambda_{\text{max}}(Q) ||b_c^2 || ||N|| ||C_b^2 || ||e_d(k)||]
(A11)

Hence condition (35) guarantees the convergence of the robust estimator to a bound dictated by the sampling period (T_s).

REFERENCES
