Higher-order sliding mode observer for estimation of tyre friction in ground vehicles

Jagat Jyoti Rath\(^1\), Kalyana Chakravarthy Veluvolu\(^1\), Michael Defoort\(^2\), Yeng Chai Soh\(^3\)

\(^1\)School of Electronics Engineering, Kyungpook National University, Daegu, South Korea
\(^2\)LAMIH, CNRS UMR 8201, Univ. Lille Nord de France, UVHC, F-59313 Valenciennes, France
\(^3\)School of Electrical and Electronic Engineering, Nanyang Technological University, Block S2, Nanyang Ave, Singapore 639798, Singapore
E-mail: veluvolu@ee.knu.ac.kr

Abstract: The estimation of friction coefficient for a vehicle when it traverses on different surfaces has been an important issue. In this work, the longitudinal vehicle dynamics, the torsional tyre dynamics and the non-linear LuGre friction dynamics are integrated to model the quarter vehicle dynamics. The road adhesion coefficient in the vehicle dynamics is unknown and varies with the contact surface. To address this issue, the authors consider a class of non-linear uncertain systems that covers the vehicle dynamics and develop a higher-order sliding mode observer based on supertwisting algorithm for state and unknown input estimations. Under Lipschitz conditions for the non-linear functions, the convergence of the estimation error is established. By estimating the road adhesion coefficient, the coefficient of friction can be estimated. Simulation results demonstrate the effectiveness of the proposed observer for state and unknown input estimation.

1 Introduction

The rapid growth of the automotive vehicle industry in recent years can be attributed to evolution of various active safety systems such as traction control systems (TCS), anti-lock breaking systems (ABS) and electronic stability programs [1–3] etc. The incorporation of these safety protocols ABS, TCS that rely on hybrid control techniques has contributed immensely to the development of intelligent automotive vehicles, that ensures smooth operation of the vehicle in changing environment. The automotive vehicle is a highly complex non-linear dynamic system, faced with severe environmental disturbances such as wind gusts, varying road profile and surface conditions. The implementation of hybrid safety protocols requires precise and robust information about the vehicle behaviour in real time. One of the major issues of vehicle dynamics concerns the friction effect between tyres and road surface, that plays a crucial role in implementation of many safety protocols.

The friction effect between tyres and road is responsible for generation of tractive force necessary for motion. This non-linear friction effect is heavily influenced by changes in the normal load, tyre pressure and changes in the road surface, and has been replicated effectively by various models such as Pacejka’s model [4], LuGre model [5] etc. In [6], the non-linear LuGre model was used for the estimation of road adhesion coefficient for a quarter-vehicle using a non-linear observer. Using the same model, non-linear adaptive observers were designed for friction estimation in [7]. Recently, in [8], the tyre road friction coefficient was estimated with a non-linear robust observer.

The use of sliding mode observers (SMO) for linear and non-linear dynamic systems has been dealt with in [9–11] for estimation of unknown inputs/faults. In [12, 13], first-order SMO-based observers were designed for estimation of faults/unknown inputs for non-linear systems. First-order sliding mode observers were designed for the estimation of internal friction state based on the LuGre friction model for applications such as servo actuators [14], pneumatic actuators [15] and three-axis turntable servo systems [16]. Based on the same LuGre model sliding mode observers were designed for the estimation of road adhesion coefficient or tyre road friction [2, 17, 18]. These methods rely on low-pass filtering to overcome chattering that is inherent in the first-order sliding mode.

To counter the inherent problems of chattering and the usage of low-pass filtering in the first-order sliding mode contributed to the development of higher-order sliding mode (HOSM) [19] theory that was devoid of chattering and hence required no low-pass filtering. The use of HOSM observers for state and unknown input estimation in uncertain non-linear systems has been discussed in [20–23]. To replace first-order SMO’s with higher-order SMO’s for estimation problems, the relative degree of the system is required to be one. For such systems, the supertwisting algorithm (STA) (see [19]), a second-order SM algorithm, was developed to provide finite time convergence in the presence of bounded perturbations. The stability proof for STA was originally shown using geometric homogeneity approaches [19] has now evolved with the use of stricter Lyapunov functions [24].
allowing the robustness of STA to be extended for a wider class of perturbations. In [25], by the development of a modified STA, class of perturbations which include linearly growing terms have been considered to improvise on the original STA. By use of Lyapunov theory, finite time convergence of the algorithm is then shown, preserving the properties of robustness and reducing the dangerous effects of chattering.

There have been several works on sliding mode to address the issues of robustness and unknown dynamics in vehicle systems. In [18], a first-order SMO based on output injection is designed for vehicle dynamics with LuGre friction model to estimate the road adhesion coefficient. This work also highlighted the advantage of sliding mode observers in comparison with the adaptive observers. An observer based on torsional tyre model to estimate the tyre friction coefficient was proposed in [26]. Based on the same model, the work in [27], designed a first-order SMO to estimate the tyre pressure and friction. However, this work did not consider the dynamic nature of friction. All these works [6, 18, 26, 27] on friction estimation did not include the non-linear longitudinal dynamics of the quarter vehicle. The works [6, 26, 27] also fell short in analysing the friction estimation problem considering its dynamic nature and surface variations. The observer-based scheme in [21] is based on a combination of high gain observer and HOSM observer. This method requires very strong structural conditions that cannot be satisfied for vehicle dynamics.

To address these issues, we consider the non-linear dynamics of the quarter vehicle and design a HOSM observer based on STA for state and unknown input estimation. The quarter vehicle dynamics in this paper is modelled by integrating the non-linear longitudinal dynamics, the torsional tyre dynamics and the non-linear LuGre friction model. The resultant dynamics is non-linear with the non-linearities satisfying the Lipschitz conditions. The road adhesion coefficient in the vehicle dynamics is generally unknown and varies with the contact surface, is treated as the unknown input. We then consider a general class of non-linear systems that covers the above non-linear dynamics to develop an HOSM observer for state and unknown input estimations. Under rank conditions for the output matrix, the non-linear system is then transformed into two subsystems where the unknown input appears in one subsystem. For the subsystem affected by the unknown input, a higher-order STA-based observer is then designed to ensure stability of the error dynamics of the subsystem in finite time. For the subsystem without unknown inputs, a non-linear observer is designed under Lipschitz conditions to ensure the stability of the system in the sliding mode. The application of the proposed method to the modelled vehicle dynamics is validated thru simulations. Comparative performance with first-order sliding mode observer highlights the advantages of the proposed method.

Throughout this paper, \(\lambda_{\text{max}}(A)\) denotes the maximum eigenvalue of the matrix \(A\), \(\|A\|\) denotes the 2-norm \(\sqrt{\lambda_{\text{max}}(A^T A)}\) of a matrix \(A\) and \(\sigma_{\text{min}}(A)\) represents the minimum singular value of matrix \(A\). For any vector \(z = [z_1, \ldots, z_i]^T \in \mathbb{R}^i\) and any scalar \(a \in \mathbb{R}\), we denote

\[
\text{sign}(z) = [\text{sign}(z_1), \ldots, \text{sign}(z_i)]^T, \\
|z|^a = \text{diag}(|z_1|^a, \ldots, |z_i|^a), \\
[z]^a = |z|^a \text{sign}(z)
\]  

where \(m\) is the mass of the vehicle, \(v\) is the longitudinal velocity, \(F_r\) is the longitudinal friction force, \(C_w\) is the drag coefficient, \(\sigma\) is the longitudinal friction force, \(g\) is acceleration due to gravity. In the considered vehicle model, the non-linear effects of the air drag force [28] and the effect of the rolling resistance are also considered. In this work, the normal load acting on the tyre is considered to be static. The equations which govern the dynamics of the tyre are given by Adcox et al. [3]

\[
J_w \ddot{w}_w = K_s (\theta_{s}) + C_t (w_r - w_w) - T_b \\
J_t \ddot{w}_t = F_r r - K_t (\theta_{s}) - C_t (w_r - w_w)
\]

where \(J_w\) and \(J_t\) are the inertia of hub and ring, respectively, \(w_w\) and \(w_r\) are the angular velocities on the hub and ring sides, respectively, \(\theta_{s}\) is the torsional angle, \(K_s\) is the spring stiffness, \(C_t\) is the damping coefficient and \(T_b\) is the braking torque.

The torsional tyre model together with the modelling parameters are shown in Fig. 1. The stiffness coefficient and the damping factor that govern the tyre properties are also depicted. The static coefficient of friction when interaction between tyre and road occurs is obtained as \(\mu_s = F_r / F_N\) (\(F_N\) is the normal load). The coefficient of friction depends on many factors which influence the interaction between the surface and tyre. Many methods (e.g. magic formula [4], Dugoff model [1]) have been proposed to model the tyre road friction coefficient. The tyre friction models such as Magic tyre model, Dugoff model are empirical functions that are based on steady-state tire force data. These models do not consider the dynamic characteristics of friction. In this work, the LuGre friction model that incorporates varying road conditions and the transient behaviour of friction during acceleration or braking has been considered. Furthermore, LuGre model dynamics also include characteristics of friction such as Stribeck effect, hysteresis, stick-slip, etc., integrated into an unified non-linear dynamic model [5]. In comparison to the Magic tyre model or the Dugoff model, the LuGre friction model is much more suited to estimate the road adhesion coefficient. The LuGre friction model...
dynamics is given as

\[
\begin{align*}
\dot{v}_r &= v - rw_r, \\
\dot{z} &= \tilde{v}_r - \frac{\sigma_1|v_i|z}{h(v_i)} \xi(t) - kr|w_r|z \\
h(v_r) &= \mu_c + (\mu_k - \mu_c) \exp^{-\frac{|v_i|}{\omega_0}}, \\
F_s &= (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_i) F_s
\end{align*}
\]  

(4)

where \(v_r\) is the relative velocity, \(r\) is the radius of tyre, \(z\) is the internal friction state, \(k\) is a constant that represents the distribution of force on a patch, \(v_i\) is the Strubeck velocity, \(\sigma_0\) is the normalised longitudinal lumped stiffness, \(\sigma_1\) is the normalised longitudinal damping, \(\sigma_2\) is the normalised viscous damping, \(\mu_c\) is the Coulomb friction coefficient and \(\xi\) is the road adhesion coefficient. In the LuGre friction model considered in [5], the distribution of the longitudinal force along the contact patch was considered to be constant. To further improve the existing model, in [29] a corrective term was introduced in the LuGre friction model (4).

In this paper, we integrate the torsional tyre model (3), LuGre dynamic friction model (4) and the longitudinal dynamics of the vehicle (2) to model the quarter vehicle. The road adhesion coefficient \(\xi(t)\) in (4) is treated as an unknown input. The integrated dynamics (2)–(4) can be represented in the state space form as

\[
\begin{align*}
\dot{x} &= A_p x + B_p(x,u) + E_p f_3(x,t) \\
y &= C_p x
\end{align*}
\]  

(5)

where

\[
x = [x_1, x_2, x_3, x_4, x_5]^T = [v, w_r, w_r, \theta_{rw}, z]^T
\]

\[
A_p = \begin{bmatrix}
    a_{11} & 0 & a_{13} & 0 & -\sigma_0 g \\
    -C_f & C_f & K_f & J_f & 0 \\
    0 & -J_f & J_f & J_f & K_f \\
    a_{31} & 0 & a_{33} & -K_f & r F_s \sigma_0 \\
    0 & -1 & 1 & 0 & 0
\end{bmatrix},
\]

\[
B_p(x,u) = \begin{bmatrix}
   f_1(x) - \sigma_1 F_s f_3(x) \\
   m \\
   -r F_s \sigma_1 \dot{f}_2(x) \\
   0 \\
   -f_2(x)
\end{bmatrix}
\]

\[
E_p = \begin{bmatrix}
   \sigma_1 \sigma_0 g \\
   0 \\
   -r F_s \sigma_1 \sigma_0 \\
   J_f \\
   -\sigma_0
\end{bmatrix}
\]

(6)

Assumption 1: \(\bullet\) \(\text{rank}(CE) = \text{rank}(E)\).

\(\bullet\) All invariant zeros of the triple \((A,E,C)\) must lie in left half plane, such that for every complex number \(\lambda\) with non-negative real part

\[
\text{rank} \begin{bmatrix}
   \lambda I - A & E \\
   C & 0
\end{bmatrix} = n + \text{rank}(E)
\]

Assumption 2: The non-linear functions in \(B(x,u)\) satisfies the Lipschitz conditions.

Assumption 3: System (10) is uniformly observable.

Assumption 4: The function \(f(x,t)\) and its derivative are bounded.

Assumption 5: The control input is bounded and the system is assumed to be bounded input bounded state stable (BIBS).

From Assumption 1, without loss of generality, we can partition \(E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}\) such that \(E_1 \in \mathbb{R}^{m \times n}\) with \(\text{rank}(E_1) = m\).
With Assumptions 1–5 satisfied, we introduce the following transformation

$$T_1 = \begin{bmatrix} I_m & 0 \\ -E_2E_1^{-1} & I_{m-n} \end{bmatrix}$$

(11)

Now we have, $CT_1^{-1} = [C_1, C_2]$, with $C_1 \in \mathbb{R}^{p \times n}$ and $C_2 \in \mathbb{R}^{p \times (m-n)}$. Using Assumption 1, it can be concluded that rank($C_1$) = $m$. Hence without loss of generality, $C_1$ and $C_2$ are partitioned as

$$C_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix}$$

where $C_{11} \in \mathbb{R}^{m \times m}$ with rank($C_{11}$) = $m$, $C_{12} \in \mathbb{R}^{(p-m) \times m}$, $C_{21} \in \mathbb{R}^{m \times (n-p)}$ and $C_{22} = \in \mathbb{R}^{(p-n) \times (n-p)}$.

Now, we have the following non-singular transformation matrices

$$T = T_1T_2, \quad T_2 = \begin{bmatrix} I_m \\ C_{11} \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & I_{m-n} \\ -C_{12}C_{11}^{-1} & I_{p-m} \end{bmatrix}$$

With the above, we can obtain the following transformed matrices

$$TATE^{-1} = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad TE = [E_1 \ 0]^T$$

$$STC^{-1} = \begin{bmatrix} C_{11} & 0 \\ -C_{12}C_{11}^{-1}C_{21} + C_{22} \end{bmatrix}, \quad \hat{x} = Tx = [\hat{x}_1 \ \hat{x}_2]^T$$

$$\hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = Sy = STC^{-1}\hat{x} = \begin{bmatrix} C_{11} \hat{x}_1 \\ (-C_{12}C_{11}^{-1}C_{21} + C_{22})\hat{x}_2 \end{bmatrix}$$

$$TB(x,u) = \Gamma(T^{-1}\hat{x},u) = \begin{bmatrix} \Gamma_1(T^{-1}\hat{x},u) \\ \Gamma_2(T^{-1}\hat{x},u) \end{bmatrix}$$

### 3.2 Observer design

For a system (10) that satisfies the above assumptions, we can design the observer of the following form

$$\hat{x} = A\hat{x} + B(\hat{x},u) + L(\gamma-C\hat{x}) + EE_1^{-1}v(t)$$

(12)

where $L$ is the feedback gain matrix to be designed later and $v(t)$ is the robust term based on the generalised STA [25] defined as

$$v(t) = -K_1\phi_1(e_1) - K_2 \int_0^t \phi_2(e_1) \, dt - K_3 e_1$$

(13)

$$\phi_1 = |e_1|^\tau \text{sign}(e_1)$$

$$\phi_2 = \text{sign}(e_1)$$

where $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ are properly designed constants. The error dynamics is defined as, $e = \hat{x} - x$ and $\bar{e} = Te = [e_1 \ e_2]^T$. Employing the transformation (11), it can be seen that the error variable $e_1$ is measurable. With the partition $S = [S_1 \ S_2]$ where $S_1 \in \mathbb{R}^{m \times p}$ and $S_2 \in \mathbb{R}^{(p-m) \times p}$, we can obtain

$$e_1 = T\bar{x}_1 - C_1 S_1 y$$

The objective of the designed observer is to ensure the error converges to zero, and to reconstruct the unknown inputs $f(x,t)$ from the robust term (13).

### 3.3 Error dynamics

For ease of analysis, we transform the error dynamics with the transformation (11) to the form

$$\dot{e}_1 = (A_{11} - H_{11})e_1 + (A_{12} - H_{12})e_2 + \Gamma_1(\hat{x},u) - \Gamma_1(x,u) + v(t) - E_1f(x,t)$$

(14)

$$\dot{e}_2 = (A_{21} - H_{21})e_1 + (A_{22} - H_{22})e_2 + \Gamma_2(\hat{x},u) - \Gamma_2(x,u)$$

(15)

where $H$ is the feedback gain matrix

$$H = \begin{bmatrix} H_{11} \\ H_{12} \end{bmatrix} = \text{TLCT}^{-1}$$

The following theorem establishes the second-order sliding mode on the surface $e_1 = \bar{e}_1 = 0$ in finite time.

**Theorem 1:** For the system (10) satisfying Assumptions 1–5, the observer system (12) with the robust term (13) will ensure that error dynamics ($e_1$) will converge to zero in finite time.

**Proof:** The transformed error dynamics (14) can be written as

$$\dot{e}_1 = v(t) + \Omega(\bar{e},t)$$

(16)

where $\Omega(\bar{e},t) = \Omega_1(\bar{e},t) + \Omega_2(\bar{e},t)$ can be separated into two components as

$$\Omega_1(\bar{e},t) = (A_{11} - H_{11})e_1$$

$$\Omega_2(\bar{e},t) = (A_{12} - H_{12})e_2 + \Gamma_1(\hat{x},u) - \Gamma_1(x,u) - E_1f(x,t)$$

(17)

The matrices $A_{11}, A_{12}, H_{11}$ and $H_{12}$ are known, and bounded. With the system satisfying Assumptions 3–5, the non-linear function $\Gamma_1(x,u)$ and the unknown input function $f(x,t)$ along with their derivatives are also bounded. We can further show that

$$\|\Omega_1(\bar{e},t)\| \leq \zeta_1\|\bar{e}_1\|$$

$$\|\Omega_2(\bar{e},t)\| \leq \zeta_2$$

(18)

It can be concluded from (18) that the perturbation terms consist of a bounded linear growing term, $\Omega_1(\bar{e},t)$ and a bounded perturbation $\Omega_2(\bar{e},t)$. With the use of modified STA [25], the compensation for linearly growing terms can be provided that was not possible earlier with the STA. As the perturbation terms (18) satisfy the conditions required for STA, the convergence of the error dynamics can now be proved similar to [25] by choosing the following Lyapunov function

$$V(e_1) = \varsigma^TQ\varsigma$$

where $\varsigma = [\phi_1(|e_1|)]^T \ e_1^T \int_0^t \phi_2(|e_1|) \, dt$ and $Q = Q^T > 0$ is the positive definite matrix defined as

$$Q = \frac{1}{2} \begin{bmatrix} 4K_2 + K_1^2 & K_1K_3 & -K_1 \\ K_1K_3 & (1 + K_2^2) & -K_2 \\ -K_1 & -K_2 & 2 \end{bmatrix}$$

The considered Lyapunov function satisfies

$$\lambda_{\min} \|\varsigma\|^2 \leq V(e_1) \leq \lambda_{\max} \|\varsigma\|^2$$

where $\lambda_{\max}$ represents the maximum eigenvalue and $\lambda_{\min}$ represents the minimum singular value. Denoting, $e_1 = \varphi_1$
and $\int_0^t \phi_2(e_1) \, dt = \psi_2$, the rate of change of the Lyapunov function can be hence obtained as

$$
\dot{V}(e) = 2\xi^TQ \xi
$$

$$
= \left[ (4K_2 + K_1^2)\psi_1^T \psi_1^2 + \psi_1^T K_1 \psi_1 - \psi_1^T K_1 \right]
\times \left[ \frac{1}{2} \text{diag}(\psi_1^2) (-K_1 \psi_1^2 + K_2 \psi_1 + \psi_2 + \Omega_1) \right]
+ \left[ K_1 \psi_1^2 + (1 + K_1^2) \psi_1^2 - K_3 \psi_1^2 \right]
\times \left[ -K_1 \psi_1^2 - K_3 \psi_1 + \psi_2 + \Omega_1 \right]
+ \left[ -K_1 \psi_1^2 + 2K_2 \psi_1^2 \right] \left[ -K_2 + \Omega_2 \right]
$$

On further simplification $\dot{V}(e)$ can be now written as

$$
\dot{V}(e) = -\frac{1}{\|\psi_1\|^2} \psi_1^T Q \psi_1 - \xi^T Q_2 \xi
$$

(19)

where (see equation at the bottom of the page)

The gains $K_1$, $K_2$, $K_3$ are designed such that $K_i > 0$; $i = 1, 2, 3$. It can be seen that the matrices $Q_1$ and $Q_2$ are positive definite for positive values of the gains and if the following inequalities are satisfied

$$
K_1 > \frac{\left( 2K_2 \zeta_1 + K_1 \zeta_2 - K_2 K_3 \right)}{2K_3 - \frac{\zeta_2}{2}}^{1/2}
$$

$$
K_2 > \frac{2\zeta_2 - K_2^2}{2}
$$

$$
K_3 > \frac{\left( K_1^2 + 2K_2 \right)}{K_2 + 2K_1^2 - \zeta_2}
$$

(20)

Thus similar to the arguments made in [25] it can be written from (19) that

$$
\dot{V} = -\frac{1}{\|\psi_1\|^2} \lambda_{\text{min}}(Q) \| \xi \|^2 - \lambda_{\text{min}}(Q_2) \| \xi \|^2
$$

Since

$$
\|\psi_1\|^2 \leq \|\xi\| \leq \frac{V^{1/2}(e)}{\lambda_{\text{min}}(Q)}
$$

$$
\dot{V} \leq \beta_1 V^{1/2}(e) - \beta_2 V(e)
$$

where

$$
\beta_1 = \frac{\lambda_{\text{min}}(Q) \lambda_{\text{min}}(Q_1)}{\lambda_{\text{max}}(Q)}, \quad \beta_2 = \frac{\lambda_{\text{max}}(Q_2)}{\lambda_{\text{max}}(Q)}
$$

With the proper selection of the gains, $K_i > 0$, $i = 1, 2, 3$, $\dot{V}(e)$ is negative definite and the error converges to zero. Thus, the sliding surface can be reached in finite time and maintained thereafter.

As error dynamics $e_i$ converges to zero in the sliding mode, we only need to establish the convergence of the reduced order error dynamics. The following theorem proves the asymptotic stability of the error dynamics $e_2$.

**Theorem 2:** For system (10) satisfying Assumptions 1–5, the observer system (12) ensures that the state estimation error is asymptotically stable provided the gain $H_{22}$ satisfies

$$
P_2(A_{22} - H_{22}) + (A_{22} - H_{22})^T P_2 + l_{v_2}^T P_2 P_2 + I < 0
$$

(21)

where $l_{v_2}$ is the Lipschitz constant for $\Gamma_3(x, u)$ and $P_2$ is a positive definite matrix.

**Proof:** In the sliding mode, the error ($e_i$) converges to zero in finite time and we have ($\dot{e}_i = e_i = 0$). The reduced-order error dynamics in the sliding mode can be obtained as

$$
\dot{e}_2 = (A_{22} - H_{22})e_2 + \Gamma_3(\hat{x}, u) - \Gamma_3(x, u)
$$

(22)

Similar to Lipschitz conditions it can be obtained

$$
\| \Gamma_3(\hat{x}, u) - \Gamma_3(x, u) \| \leq l_{v_2} \| e_2 \|
$$

(23)

for some Lipschitz constant $l_{v_2}$. By the choice of Lyapunov candidate function $V_2 = e_2^T P_2 e_2$, and differentiating (22) with respect to time, one can obtain

$$
\dot{V}_2 = e_2^T P_2 \dot{e}_2 + \dot{e}_2^T P_2 e_2
$$

$$
= e_2^T \left[ (A_{22} - H_{22})^T P_2 + P_2 (A_{22} - H_{22}) \right] e_2
$$

$$
+ 2e_2^T P_2 \Gamma_3(\hat{x}, u) - \Gamma_3(x, u)
$$

From the above results, it can be deduced

$$
\dot{V}_2 \leq e_2^T \left[ (A_{22} - H_{22})^T P_2 + P_2 (A_{22} - H_{22}) \right] e_2
$$

$$
+ 2\| P_2 e_2 \| \| e_2 \| l_{v_2}
$$

$$
\leq \beta_1 V^{1/2}(e) - \beta_2 V(e)
$$

$$
Q_1 = \begin{bmatrix}
(2K_2 + K_1^2) \frac{K_1}{2} - K_1 \zeta_2 & 0 & 0 & -K_1^2 \\
0 & K_1 \left( \frac{K_1}{2} + 5K_1^2 \right) & -3K_1 \zeta_1 & -3K_1 \zeta_3 \\
-\frac{K_1^2}{2} & -3K_1 \zeta_1 & K_1 \frac{K_1}{2} & -K_1 \zeta_3 \\
(2K_2 + K_1^2) \zeta_3 - K_3 \zeta_2 & 0 & -K_1 \zeta_1 & K_1 \zeta_3
\end{bmatrix}
$$

$$
Q_2 = \begin{bmatrix}
0 & 0 & -K_1 \zeta_1 & -K_1 \zeta_1 \\
0 & K_1 \left( \frac{K_1}{2} + 1 \right) & -K_1 \zeta_1 & -K_1 \zeta_3 \\
-K_1 \zeta_1 & -K_1 \zeta_3 & K_1 \frac{K_1}{2} & -K_1 \zeta_3 \\
-K_1 \zeta_1 & -K_1 \zeta_3 & K_1 \zeta_3 & K_1 \zeta_3
\end{bmatrix}
$$
As
\[ 2l_{r_1}\|P_2e_2\|\|e_2\| \leq l_{r_2}e_1^T P_2^2 P_2 e_2 + e_1^T e_2 \]
one can obtain
\[
\dot{v}_2 \leq (A_{22} - H_{22})^T P_2 + P_2 (A_{22} - H_{22}) \\
+ (l_{r_1} e_1^T P_2 P_2 + 1) e_2
\]
If the feedback gain \( H \) is designed such that (21) is satisfied, then \( \dot{v}_2 < 0 \). Thus the error dynamics in the sliding mode will be asymptotically stable. \( \Box \)

The gain \( H_{22} \) is designed such that (21) is satisfied. This equation can be written as an algebraic Riccati equation of the form
\[
P_2 (A_{22} - H_{22}) + (A_{22} - H_{22})^T P_2 + (l_{r_1} e_1^T P_2 P_2 + 1) e_1 = 0
\]
for some \( \epsilon > 0 \). The following condition, according to [31, 32] ensures the asymptotic stability of the system (22)
\[
\min_{\mathcal{R}^n} \sigma_{\text{min}} (A_{22} - H_{22} - j \omega I) > l_{r_2}
\]
where \( \sigma_{\text{min}}(\cdot) \) represents the minimum singular value of a matrix. If the above condition (21) is satisfied and if there exists a stable \( (A_{22} - H_{22}) \) matrix, then there exists a symmetric positive definite (SPD) solution \( P_2 = P_2^T \) for Riccati equation (24).

Further, the Ricatti equation (24) can be transformed into a matrix inequality problem as follows
\[
\begin{bmatrix}
P_2 (A_{22} - H_{22}) + (A_{22} - H_{22})^T P_2 + (l_{r_1} e_1^T P_2 P_2 + 1) e_1 & 1 \\
1 & -1
\end{bmatrix} < 0
\]
(25)

Remark 3: The stability of the error dynamics is established in two stages. During the initial stage when \( e_1 \to 0 \), \( e_1 \) remains bounded. As \( e_1 \) remains bounded during the transient, the boundedness of \( e_1 \) can be easily established similar to Theorem 2. The presence of term \( e_1 \) in the dynamics of \( e_2 \) because of coupling will ensure the ultimate boundedness for the error dynamics \( e_2 \). In the second stage, after \( e_1 \to 0 \) in finite time, we have \( e_1 = 0 \). The reduced-order dynamics \( e_2 \) is asymptotic stable in the sliding mode of \( e_1 = 0 \) as shown in Theorem 2. The proposed design guarantees finite time stability for \( e_1 \) and asymptotic stability for \( e_2 \).

Remark 4: If the number of unknown inputs is equal to number to outputs (i.e. \( m = p \)), then the reduced order dynamics should be stable as there is no extra output to stabilise the reduced order system. However, for non-linear systems the reduced order dynamics may not be stable because of the presence of non-linear functions. Thus to establish the stability of the reduced order dynamics, extra outputs \( (p > m) \) are required. The remaining outputs \( (p - m) \) are used to stabilise the reduced order system with the feedback gain \( H_{22} \). As all the states are measurable as outputs in the \( e_1 \) sub-system, one can design the gain \( H_{12} \) such that \( (A_{12} - H_{12}) \) is stable. Without loss of generality, one can further select \( H_{12} = H_{21} = 0 \).

3.4 Unknown input reconstruction

From (14), with error \( (e_1) \) converging to zero in finite time \( (\dot{e}_1 = e_1 = 0) \), we have
\[
v_{eq} = -\Omega (\dot{e}, t) + E_i f(x, t)
\]
where
\[
\Omega (\dot{e}, t) = (A_{11} - H_{11}) e_1 + (A_{12} - H_{12}) e_2 \\
+ \Gamma_1 (\dot{x}, u) - \Gamma_1 (x, u)
\]
As \( e_1 \to 0 \) in finite time and the non-linearities satisfy Lipschitz assumptions we have
\[
\|\Omega (\dot{e}, t)\| \leq (\|A_{12} - H_{12}\| + l_{r_2}) \|\dot{e}(t)\| \to 0 (t \to \infty)
\]
The equivalent output error injection signal can now be obtained from the generalised STA as
\[
v_{eq} = K_2 \int_{0}^{\tau} \text{sign}(e_1(t)) \, dt
\]
(26)
Hence, the unknown input when \( t \to \infty \) can be estimated as
\[
f(x, t) = E_i^{-1} K_2 \int_{0}^{\tau} \text{sign}(e_1(t)) \, dt
\]
(27)

4 HOSM observer design for the vehicle system

As the longitudinal velocity, \( x_1 \) and the angular velocity on the ring side, \( x_3 \) are considered as outputs, without loss of generality we chose the outputs as
\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_p x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ mr & 0 & 1 & 0 & 0 \end{bmatrix} x
\]
(28)
For the dynamic vehicle system in (5), we have \( \text{rank}(C_p E_p) = \text{rank}(E_p) = 1 \). Therefore Assumption 1 is satisfied. The matrix \( E_p \) is thus partitioned as, \( E_p = [E_{p1} E_{p2}] \), \( E_{p1} = \sigma_1 \sigma_0 g \) and \( E_{p2} = [0 - \sigma_0 \sigma_0 - \sigma_0] \). The transformation matrix can be obtained according to (11) as
\[
T = \begin{bmatrix} 1 & 0 & rm & 0 & 1 \\ 0 & \sigma_0 \sigma_0 & -1 & 0 & \sigma_0 \end{bmatrix}
\]
(29)
We also have \( C_{p11} = 1 \) and \( C_{p22} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \). The HOSM observer (12) can be designed for the system (5) as
\[
\dot{\hat{x}} = A_p \hat{x} + B_p (\hat{x}, u) + L (y - C_p \hat{x}) + E_p E_p^{-1} v(t)
\]
(28)
\[
v(t) = - \left( K_1 \phi_1 (\hat{x}_1 - y_1) + K_2 \int_{0}^{t} \phi_2 (\hat{x}_1 - y_1) \, dt \right)
\]
(29)
of the system can now be obtained as

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3 \\
\dot{e}_4 \\
\dot{e}_5
\end{bmatrix} = \begin{bmatrix}
e_{11} & \ldots & e_{15} \\
e_{21} & \ldots & e_{25} \\
e_{31} & \ldots & e_{35} \\
e_{41} & \ldots & e_{45} \\
e_{51}
\end{bmatrix} = \begin{bmatrix}
e_{11} & \ldots & e_{15} \\
e_{21} & \ldots & e_{25} \\
e_{31} & \ldots & e_{35} \\
e_{41} & \ldots & e_{45} \\
e_{51}
\end{bmatrix}
\]

where \( e_{11} = \sigma_0/\sigma_1 - [(FrN^2(\sigma_1 + \sigma_2)/Jr) - l_{14} - \sigma_2 + \sigma_0] \), \( e_{15} = - [(FrN^2(\sigma_1 + \sigma_2)/Jr) - l_{25} - \sigma_0] \), \( e_{35} = - [(FrN^2(\sigma_1 + \sigma_2)/Jr) - l_{35} - \sigma_0] \), \( e_{45} = - [(FrN^2(\sigma_1 + \sigma_2)/Jr) - l_{45} - \sigma_0] \), and \( e_{51} = e_{51} \). Representing the transformed error dynamics of the system as in (16), for the above vehicular system, the bound constants \( \xi_1 \) and \( \xi_2 \) can be easily obtained. The non-linear functions \( f_1(x) \) and \( f_2(x) \) in the vehicle dynamics (5) are Lipschitz [6] and hence bounded. The Lipschitz constants can be obtained from [6] for \( f_1(x) \), \( f_2(x) \) and \( f_3(x) \) as \( K_{\xi} = C_{\sigma_0} |x|_{\max} \) and \( K_{\varepsilon} = C_{\sigma_0} |x|_{\max} \), respectively, where \( K_{\xi} = \|x\|_{\max}/\mu_{\xi} \) and \( K_{\varepsilon} = \|x\|_{\max}/\mu_{\xi} \). Hence, Assumptions 2-4 are satisfied. As the input \( T_0 \) is known and bounded, Assumption 5 is satisfied. Once the sliding mode is established, we can recover the unknown input as

\[
v_{eq} = -\Omega(\tilde{x}, t) + g \sigma_0 \sigma f_3(x, t)
\]

where \( \Omega(\tilde{x}, t) = e_{11} e_1 + [0 0 -\sigma_2 \sigma_0 e_{21} + f_{31}]. \) We have \( \Omega(\tilde{x}, t) \to 0 \) as \( t \to \infty \). The unknown input can be reconstructed from the sliding mode similar to (26) as

\[
\dot{\tilde{v}}(t) = K_{\sigma_0} \int_0^t \hat{\phi}_2(\tilde{v}(t), t) dt
\]

As the estimated states converge to the true states, the road adhesion coefficient can be approximated from \( \dot{\tilde{v}}(t) \) as follows

\[
\hat{\xi}(t) = \frac{\dot{\tilde{v}}(t)}{\int_0^t e_1(t) dt}
\]

Remark 5: The estimated road adhesion parameter \( \xi(t) \) in (31) has a singularity when the relative velocity between the tires and the vehicle velocity is zero that is, \( \xi(t) = 0 \) or when the internal friction state is zero, that is, \( \xi = 0 \). Under such conditions, traction that is dependent on friction, is lost. To avoid this scenario, the vehicle in general is assumed to be either under acceleration or deceleration such that \( \eta \) never approaches zero that in turn does not let \( z \) approach to zero [6]. For low relative velocity, that is, when \( \eta \) is zero, the estimation will be less accurate.

4.1 Simulation results

The parameters for the quarter vehicle model are chosen according to [3, 18] as follows: vehicle mass, \( M = 2040 \text{ kg} \), quarter vehicle mass, \( M = M/4 = 510 \text{ kg} \), \( C_r = 88 \text{ Nm/s} \), \( K_r = 1.65 \times 10^6 \text{ N/m} \), \( r = 0.3 \text{ m} \), \( J_r = 1 \text{ kgm}^2 \), \( J_r = 0.093 \text{ kgm}^2 \), \( \sigma_r = 0.005 \), \( \sigma_r = 100 \text{ m/s} \), \( \sigma_r = 0.7 \text{ m/s} \), \( \mu_r = 0.35 \), \( \mu_r = 0.5 \), \( r = 0.315 \text{ m} \) and \( k = 5.83 \text{ m}^{-1} \). The initial conditions for the plant and observer were chosen as \( x(0) = [30 10 10 0 0] \) and \( \dot{x}(0) = [25 10 10 0 0.001] \). The Lipschitz constant for the reduced system has been evaluated as \( l_1 = 1.764 \). The feedback gain matrix is selected as

\[
L = \begin{bmatrix}
-184.4152 & 2 & 3 & 3 & -0.0023
\end{bmatrix}^T
\]

The positive definite matrix \( P_2 \) that satisfies (21) can be obtained as

\[
P_2 = \begin{bmatrix}
0.0002 & 0.044 & 0.0481 & 0 & 0.0094 & 0.0481 & 0 & -0.004 & 26.68 & 0
\end{bmatrix}
\]

To obtain the bound of \( \xi_z \), the rate of change of the disturbance \( \xi_z \) can be obtained from (15) and (17) as

\[
\dot{\xi}(t) = (A_{12} - H_{12}) \xi_z + \frac{\partial}{\partial t} \Gamma_1(\tilde{v}, \tilde{x}) - \frac{\partial}{\partial t} \Gamma_1(\tilde{v}, \tilde{x})
\]

During the initial stage of error convergence, the error dynamics \( \xi_z \) and \( \xi_z \) will be bounded (Remark 3). The derivative of the unknown input function and non-linear function \( \Gamma_1(\tilde{v}, \tilde{x}) \) in (32) can be obtained as

\[
\frac{\partial}{\partial t} \Gamma_1(\tilde{v}, \tilde{x}) = \frac{2C_{\sigma_0}}{m} \left| \hat{v}(t) \right| + \sigma_0 \sigma_0 \sigma f_3(x, t) - \frac{\sigma_0 \sigma_0 \sigma f_3(x, t)}{J_r} - \frac{\sigma_0 \sigma_0 \sigma f_3(x, t)}{J_r} - \frac{\sigma_0 \sigma_0 \sigma f_3(x, t)}{J_r}
\]

Based on Assumption 5, the states remain bounded as input is bounded. As error dynamics is bounded, we have the estimated states \( \hat{x} \) also bounded. The derivatives of the functions \( f_1(x, t) \), \( \Gamma_1(\tilde{v}, \tilde{x}) \) and \( \Gamma_1(\tilde{v}, \tilde{x}) \) depends on the known system parameters, system states (x) and estimated states (\( \hat{x} \)). Based on above arguments, the bound for \( \xi_z(\tilde{v}, \tilde{x}) \) can be a priori selected. For the above parameters, the bounds are set as, \( \xi_z = 0.1 \) and \( \xi_z = 10.68 \). Notice that this upper bound is verified posteriori through numerical simulations (this procedure is usually employed in higher-order sliding mode applications [33]). The observer gains are chosen according to conditions in (20) as \( K_{11} = 1, K_{12} = 25 \) and \( K_{11} = 0.31 \).

The simulation has been performed for low slip velocities. To simulate different road conditions, \( \xi(t) \) for different road surfaces snow, wet and dry asphalt were chosen as 4, 2.5
Fig. 2  Actual and estimated states with proposed observer

\(a\)  \(x_1\)
\(b\)  \(x_2\)
\(c\)  \(x_3\)
\(d\)  \(x_4\)
\(e\)  \(x_5\)

Fig. 3  Estimated error and unknown input

\(a\) Error norm \(\|e\|\)
\(b\) Estimated unknown input using proposed observer

and 1, respectively [6]. The unknown input \(f_1(x, t)\) and the road adhesion coefficient \(\xi(t)\) can be estimated using (30) and (31).

In Fig. 2, the performance of the proposed observer in the presence of the unknown input \(\xi(t)\) is shown. Despite the variation in \(\xi(t)\) for changes in road surface, the observer tracks the respective states accurately. The norm of the error dynamics is shown in Fig. 3a. The reconstruction the unknown input \(f_1(x, t)\) from the sliding mode as shown in Fig. 3a. The road adhesion coefficient obtained from (31) is shown in Fig. 4. It can be clearly seen that the proposed method can provide chattering free and finite time estimation.
of $\xi(t)$. To further test the robustness, 5% random noise is added to the output measurements. The estimation of $\xi(t)$ in the presence of noise is shown in Fig. 4.

A comparative analysis is performed between the proposed higher-order SMO and the first-order SMO for the estimation of $\xi(t)$ and the results are shown in Fig. 5. To obtain the estimation of the unknown input $\xi(t)$ with first-order SMO a fifth order butterworth low-pass filter is employed. The usage of this additional filter introduces an unwanted delay into the estimation process. It can be seen from Fig. 5 that even after filtering, the chattering is not completely eliminated. In comparison, the proposed STA provides chattering free robust estimation of $\xi(t)$.

![Estimated road adhesion coefficient](image)

**Fig. 4** Estimated road adhesion coefficient

*a* Estimation of $\xi(t)$

*b* Estimation of $\xi(t)$ in presence of 5% noise in output using proposed observer

![Estimation of road adhesion coefficient](image)

**Estimation of road adhesion coefficient**

- $\hat{\xi}$ (First order)
- $\hat{\xi}$ (Second order)
- $\xi$ (Reference)

![Estimated road adhesion coefficient](image)

**Fig. 5** Estimation of $\xi(t)$ with proposed observer and first-order SMO

To obtain the estimation of the unknown input $\xi(t)$ with first-order SMO a fifth order butterworth low-pass filter is employed. The usage of this additional filter introduces an unwanted delay into the estimation process. It can be seen from Fig. 5 that even after filtering, the chattering is not completely eliminated. In comparison, the proposed STA provides chattering free robust estimation of $\xi(t)$.

5 Conclusion

In this work, a SMO based on STA was proposed for state and unknown input estimations with application to a quarter vehicle. The torsional tyre model and the non-linear LuGre model are integrated for modelling the quarter vehicle. A higher-order sliding mode observer is designed for estimation of states and unknown inputs. Under the Lipschitz condition for the non-linear functions, the convergence of the reduced order system in the sliding mode is proven. The proposed higher-order SMO accurately estimates the road adhesion coefficient under various surface conditions without the use of low-pass filter. Comparison of the proposed observer with a first-order SMO highlights the advantages.

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7 References


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