Adaptive sensor and actuator fault estimation for a class of uncertain Lipschitz nonlinear systems

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SUMMARY

This paper deals with state and fault estimations for a class of uncertain Lipschitz nonlinear systems. The proposed scheme combines a descriptor form observer and an adaptive sliding mode observer. The adaptive descriptor sliding approach is proposed to jointly estimate the state and reconstruct the sensor fault while rejecting the effect of uncertainties and Lipschitz nonlinearities. Using a Lyapunov analysis, the stability condition is analyzed and the observer gains are designed such that the reduced–order system is practically stable. Then, an adaptive super-twisting observer is derived to reconstruct the actuator faults. The main feature of the proposed adaptive scheme is that it does not overestimate the observer gains, which mitigates chattering in the presence of bounded uncertainties, actuator, and sensor faults with unknown boundaries. Simulation results for a robotic manipulator show the effectiveness of the proposed method. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Occurrence of faults can lead to critical situations for systems (instability, deterioration, etc.) and can be extremely detrimental, not only to the equipment and surroundings but also to the human operator if they are not detected and isolated in time. Fault detection and isolation (FDI) and fault tolerant control have been widely investigated using various methods [1–6]. Observer-based FDI techniques rely on the estimation of outputs from measurements with the observer in order to detect the fault. They are usually based on either the analysis of appropriate residuals (i.e., the comparison between output estimations and the measurements) or an estimation of the fault variables. Most of the existing frameworks on fault estimation can be combined with unknown input estimation problem in the literature. For the case of linear systems, the unknown input estimation has been well understood and several results are available [2, 7–10]. The work in [8] extended the earlier research of [2] for the case of nonlinear Lipschitz bounded systems using a sliding mode observer (SMO). The work in [11] developed an observer for unknown input estimation that requires the evaluation of output higher-order derivatives. However, in real-world applications, it is not easy to directly apply the existing schemes because of the presence of uncertainties, disturbances, and noise [12, 13].

Sliding mode observers have been developed to robustly estimate the system states through the concept of sliding surface design and equivalent control. This discontinuous technique consists in

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constraining the system motion along manifolds of reduced dimensionality in the state space and is applicable to a broad variety of practical applications. Robustness properties against various kinds of uncertainties such as parameter perturbations and external disturbances can be guaranteed.

Several SMO-based FDI approaches have been developed recently for linear systems and a class of nonlinear systems. The first-order SMOs in [14–16] are designed to directly tackle the actuator faults through the sliding mode terms via some transformations. Unfortunately, the realization of first-order sliding mode implies the undesirable chattering phenomenon, that is, high-frequency vibrations of the closed-loop system, which may excite unmodeled high-frequency dynamics, degrade the system performances, and lead to instability. Other techniques rely on a nonlinear coordinate transformation to obtain a form suite for the design of high-gain observer [17, 18]. The gains are designed based on the inverse jacobian of the state transformation. However, this transformation is valid only locally and no sensor fault is considered. Recently, some SMOs have been proposed to reconstruct both the system state, actuator faults, and sensor faults. In [19, 20], actuator fault is reconstructed with the so-called equivalent output estimation error injection concept. Then, sensor fault is estimated via a low-pass filter and appropriate secondary SMO. Nevertheless, the sensor fault estimation is very sensitive to the filter parameters, and no uncertainty acting on the system is considered. An SMO is proposed in [21, 22] to solve this problem for nonlinear systems without uncertainty assuming some conditions on the fault distribution matrix. Descriptor sliding mode approaches are introduced in [5, 23] to simultaneously reconstruct state, sensor, and actuator faults for linear systems. In [24], a sliding mode unknown input observer is given for linear systems where the same additive perturbation affects both the dynamics and the output. In [25], an \( H_\infty \) SMO is derived for a class of uncertain nonlinear Lipschitz systems with sensor and actuator faults. Most of the mentioned SMOs require the knowledge of the upper bounds of the unknown input and uncertainties. Anyway, in practise, these bounds are usually difficult to obtain due to the complexity and unpredictability of the system uncertainties.

Recently, adaptive sliding mode schemes have been proposed to avoid a gain overestimation. This gain adaptation induces a lower chattering phenomenon. A dynamic gain adaptation for first-order sliding mode has been given in [26] for a class of nonlinear systems with bounded uncertainties whose bounds are unknown. The adaptation algorithm ensures that the gain is not overestimated, which leads to a reduction of chattering. Recently, it has been extended to second-order sliding mode control [27]. Similarly to [26], the controller only guarantees the real 2-sliding mode, that is, the sliding surface and its first time derivative converge to a known neighborhood around the origin.

Bounded-error or set-membership methods assume that the system is affected by a bounded error [28]. These methods allow the characterization of the set of state values that is consistent with the measurement, the model structure and the prior knowledge of the error bounds [29]. The state estimation based on bounded uncertainty assumption is called set-valued observer [30]. This scheme has been applied in several fault detection algorithms [31].

The objective of this paper is the robust state and fault estimation for uncertain Lipschitz nonlinear systems with simultaneous unknown input and possible sensor fault. Here, the upper-bounds of the uncertainties, unknown inputs, and sensor faults are not a priori known. It should be highlighted that the adaptive SMO design for uncertain nonlinear systems with actuator and sensor faults with unknown boundaries remains an open problem. Through a series of transformations, the original system is transformed into two subsystems. One subsystem is transformed into a descriptor form by considering the sensor fault and the uncertainties as supplementary state variables. An adaptive descriptor sliding approach is proposed to jointly estimate the state and reconstruct the sensor fault while rejecting the effect of uncertainties and Lipschitz nonlinearities. The observer gains are designed by solving some LMI conditions to guarantee the practical stability of the error dynamics. Then, using the adaptation scheme proposed in [27], an adaptive super-twisting observer is used to reconstruct the actuator faults. The contribution of our paper lies in the following features: (1) an accurate estimation of the state, the sensor and actuator faults is provided in spite of the presence of uncertainties and Lipschitz nonlinearities, (2) conditions on the observer gains are given in terms of LMI; and (3) the upper bounds of the uncertainties, sensor, and actuator faults
are not a priori known. Simulation results for a robotic manipulator show the effectiveness of the proposed method.

The paper is organized as follows. The problem statement is described in Section 2. In Section 3, the adaptive observer is presented to estimate the state, the sensor and actuator faults in spite of the presence of uncertainties and Lipschitz nonlinearities. An application to a robot-arm problem is provided in Section 4. Section 5 concludes the paper.

1.1. Notation
Throughout this paper, $\lambda_{\text{max}}(A)$ (resp. $\lambda_{\text{min}}(A)$) denotes the maximum (resp. minimum) eigenvalue of a matrix $A$. For a symmetric matrix, $A > 0$ means that $A$ is a positive definite matrix. $\| \cdot \|$ denotes the Euclidean norm of a vector.

For any vector $x = [x_1, \ldots, x_n] \in \mathbb{R}^n$ and any scalar $a \in \mathbb{R}$, we denote

$$\text{sign}(x) = [\text{sign}(x_1), \ldots, \text{sign}(x_n)]^T$$

$$|x|^a = \text{diag}(|x_1|^a, \ldots, |x_n|^a)$$

$$|x|^a = |x|^a \text{sign}(x)$$

2. PROBLEM STATEMENT
Consider the following system

$$\begin{cases}
\dot{x}(t) = Ax(t) + B\Phi(x(t), u(t)) + E f_a(t) + \Delta \xi(t) \\
y(t) = Cx(t) + Df_s(t)
\end{cases}$$

(1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times s}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m_1}$, $E \in \mathbb{R}^{n \times m_1}$, $\Delta \in \mathbb{R}^{n \times r}$ with $x \in \mathcal{M} \subset \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subset \mathbb{R}^q$ ($\mathcal{U}$ is an admissible control set) is the control input and $y \in \mathcal{Y} \subset \mathbb{R}^p$ ($\mathcal{Y}$ is the output space) is the system output. The unknown function $f_a \in \mathbb{R}^{n_1}$ is the actuator fault. The signal $\xi \in \mathbb{R}^r$ models the parameter uncertainties and/or disturbances. The sensor fault is $f_s \in \mathbb{R}^{m_2}$. The function $\Phi(x, u) : \mathbb{R}^{n+q} \mapsto \mathbb{R}^s$ is Lipschitz about $x$ uniformly, that is,

$$\|\Phi(x_1, u) - \Phi(x_2, u)\| \leq L_\Phi \|x_1 - x_2\|$$

(2)

where $L_\Phi$ is the known Lipschitz constant.

The objective of this paper is to design a fault estimation scheme for the uncertain system (1) such that accurate estimates of $x$, $f_s$ and $f_a$ are provided. To facilitate the design of the observer, the following assumptions are required:

Assumption 1

(a) $D$, $E$, and $\Delta$ are full-column rank
(b) The system matrix dimensions satisfy $m_1 + m_2 + r \leq p$
(c) For every complex number $s'$ with nonnegative real part: $\text{rank}
\begin{bmatrix}
sI_n - A & E \\
C & 0
\end{bmatrix}
= n + m_1.$

It is equivalent to the minimum phase condition. Furthermore, there exists $\delta > 0$ such that

$$\text{rank}
\begin{bmatrix}
\delta I_n + A & E & \Delta \\
C & 0 & 0
\end{bmatrix}
= n + m_1 + r$$

(d) The detectability condition holds, that is, $\text{rank}(CE) = \text{rank}(E)$.

Remark 1

Assumption 1 enforces structural limitations and pertains to the dimensional aspects of the nonlinear system (1). One can notify that condition 1(b) can be relaxed if not all outputs are affected by sensor faults or if all the outputs are affected by the same sensor fault.
assess the geometric condition $Im(E) \cap Im(\Delta) = \{0\}$ holds. This assumption implies that the image/column space of matrices $E$ and $\Delta$ are independent under any linear transformation. Roughly speaking, it induces a decoupling scheme between actuator faults and uncertainties.

**Assumption 3**

The unknown functions $f_a, \xi, f_s$, and their first-time derivative are norm bounded. However, the upper bounds of the uncertainties, actuator, and sensor faults are not a priori known.

From Assumption 1, without loss of generality, we can assume the following geometric condition associated with $D$, that is, $D = \begin{bmatrix} 0 \\ D_2 \end{bmatrix}$ where $D_2 \in \mathbb{R}^{p-m_1 \times m_2}$ is full-column rank. $E$ is partitioned as $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ such that $E_1 \in \mathbb{R}^{m_1 \times m_1}$ with $\text{rank}(E_1) = m_1$. Let us introduce the nonsingular transformation [25]

$$ T_1 = \begin{bmatrix} I_{m_1} \\ -E_2 E_1^{-1} I_{n-m_1} \end{bmatrix} $$

$CT_1^{-1}$ is partitioned as $CT_1^{-1} = [C_1 \ C_2]$ such that $C_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} \in \mathbb{R}^{p \times m_1}$ and $C_2 = \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} \in \mathbb{R}^{p \times (n-m_1)}$ with $\text{rank}(C_{11}) = m_1$. Considering the nonsingular transformation matrices [25]

$$ T = T_2 T_1 \ , \ T_2 = \begin{bmatrix} I_{m_1} \\ 0 \\ C_1^{-1} C_{21} \end{bmatrix} \ , \ H = \begin{bmatrix} I_{m_1} \\ -C_{12} C_{11}^{-1} \\ I_{p-m_1} \end{bmatrix} $$

we have

$$ TAT^{-1} = \begin{bmatrix} A_1 \\ A_3 \\ A_4 \end{bmatrix} \ , \ TE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \ , \ HCT^{-1} = \begin{bmatrix} C_{11} \\ 0 \\ C_3 \end{bmatrix} \ , \ C_3 = -C_{12} C_{11}^{-1} C_{21} + C_{22} \\
HD = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \ , \ TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \ , \ T\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} $$

From Assumption 2, using the transformation $T$, one gets $Im(TE) \cap Im(T\Delta) = \{0\}$. From the nonsingularity of $E_1$, it follows that $\Delta_1 = 0$.

Therefore, system (1) in the new coordinates $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T x \ , \ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = H y$ is

$$ \begin{aligned}
\dot{x}_1 &= A_1 x_1 + A_2 x_2 + B_1 \phi(x_1, x_2, u) + E_1 f_a \\
y_1 &= C_{11} x_1 
\end{aligned} $$

(5)

$$ \begin{aligned}
\dot{x}_2 &= A_3 x_1 + A_4 x_2 + B_2 \phi(x_1, x_2, u) + \Delta_2 \xi \\
y_2 &= C_{3} x_2 + D_2 f_s 
\end{aligned} $$

(6)

with $\phi(x_1, x_2, u) = \Phi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T u$. From Equation (2), one can conclude that

$$ \|\phi(x_1, \bar{x}_2, u) - \phi(x_1, x_2, u)\| \leq L_\phi \|\bar{x}_2 - x_2\| $$

(7)

where $L_\phi$ is the corresponding Lipschitz constant.

Based on the transformed system (5)–(6), an estimation scheme is proposed such that accurate estimates of $x, f_s$ and $f_a$ are obtained.
3. SLIDING MODE OBSERVER DESIGN

The fault estimation strategy for system (1) is based on two steps. First, system (6) is transformed into a descriptor form. An adaptive descriptor sliding approach is proposed to jointly estimate the state $x_2$ and reconstruct the sensor fault $f_s$ while rejecting the effect of uncertainties and Lipschitz nonlinearities. The observer gains are designed by solving some linear matrix inequalities conditions to guarantee the practical stability of the error dynamics. At last, from system (5), using the adaptation scheme proposed in [27], an adaptive super-twisting observer is used to reconstruct the actuator faults $f_a$.

3.1. Descriptor adaptive observer for state and sensor fault estimation

First, let us consider subsystem (6). It will be shown hereafter that it can be transformed into a descriptor form. Then, an adaptive descriptor sliding approach will be proposed to jointly estimate the state $x_2$ and reconstruct the sensor fault $f_s$. Some conditions on the observer gains, based on linear matrix inequalities, will be given to guarantee the practical stability of the error dynamics.

Because $C_{11}$ is invertible, from subsystem (5), one gets $x_1 = C_{11}^{-1} y_1$. Hence, subsystem (6) can be rewritten as follows:

$$
\begin{align*}
\dot{x}_2 &= A_3 C_{11}^{-1} y_1 + A_4 x_2 + B_2 \phi (C_{11}^{-1} y_1, x_2, u) + \Delta_2 \xi \\
y_2 &= C x_2 + D_2 f_s
\end{align*}
$$

(8)

Because Assumption 1(c) holds, the pair $(A_4, C_3)$ is detectable. This preliminary assumption is required to estimate $x_2$ from subsystem (6). System (8) can take the following descriptor form

$$
\begin{align*}
T \ddot{x} &= \tilde{A}_4 \ddot{x} + \tilde{B}_1 y_1 + \tilde{B}_2 \phi (C_{11}^{-1} y_1, x_2, u) + \tilde{\Delta} \xi \\
y_2 &= \tilde{C} \ddot{x}
\end{align*}
$$

(9)

where the extended state is $\ddot{x} = [x_2^T \xi^T (D_2 f_s)^T]^T$ and

$$
\tilde{A}_4 = \begin{bmatrix}
A_4 & 0 & 0 \\
0 & -\delta I_r & 0 \\
0 & 0 & -I_{p-m_1}
\end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix}
A_3 C_{11}^{-1} \\
0 \\
0
\end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix}
B_2 \\
0 \\
0
\end{bmatrix}, \quad \tilde{C} = \begin{bmatrix}
C_3 & 0 & I_{p-m_1}
\end{bmatrix}
$$

$$
T = \begin{bmatrix}
I_{m-1} & \delta^{-1} \Delta_2 & 0 \\
0 & I_r & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \tilde{\Delta} = \begin{bmatrix}
\delta^{-1} \Delta_2 & 0 \\
0 & I_r & 0 \\
0 & 0 & D_2
\end{bmatrix}, \quad \ddot{x} = \begin{bmatrix}
\delta \xi + \xi \\
\xi
\end{bmatrix}
$$

Let us first introduce the following two lemmas that will be used in the derivation of the adaptive observer.

**Lemma 1** ([5])

There exists a matrix $L_1$ such that the matrix $S = T + L_1 \tilde{C}$ is nonsingular. Furthermore, it holds that $C S^{-1} L_1 = I_{p-m_1}$ and $A_4 S^{-1} L_1 = -G$ with $G = \begin{bmatrix} 0 & 0 & I_{p-m_1} \end{bmatrix}^T$.

**Lemma 2** ([5])

If $\text{rank} \left( \begin{bmatrix} \delta I_{n-m_1} + A_4 & \Delta_2 \\ C_3 & 0 \end{bmatrix} \right) = n - m_1 + r$, there is a gain matrix $L_2$ such that the matrix $S^{-1} (\tilde{A}_4 - L_2 \tilde{C})$ is Hurwitz.

Using Assumption 1(c), one can easily show that, $\text{rank} \left( \begin{bmatrix} \delta I_{n-m_1} + A_4 & \Delta_2 \\ C_3 & 0 \end{bmatrix} \right) = n - m_1 + r$.

Hence, one can apply Lemma 2.

From Assumption 3, it follows that $\ddot{\xi}$ is bounded by an a priori unknown constant, that is, $\ddot{\xi} \leq \Lambda$. This constant is not a priori known.
Let us introduce the descriptor SMO

\[
\begin{align*}
S\dot{z} &= (\tilde{A}_4 - L_2 \tilde{C})z - G y_2 + \tilde{B}_1 y_1 + \tilde{B}_2 \phi(C^{-1}_{11} y_1, \hat{x}_2, u) + k \tilde{B}_2 M(y_2 - \tilde{C} \hat{x}) + \Delta v_1(\tilde{e}) \\
\hat{x} &= z + S^{-1} L_1 y_2
\end{align*}
\]

where \( z \) is an auxiliary variable, \( \hat{x} = \begin{bmatrix} \hat{x}_2^T & \hat{\xi}^T & \hat{\xi}_s^T \end{bmatrix}^T \) is the estimation of \( \hat{x} \) and the corresponding estimation error is

\[
\tilde{\epsilon} = \hat{x} - \hat{x}
\]

The term \( \tilde{B}_2 \phi(C^{-1}_{11} y_1, \hat{x}_2, u) + k \tilde{B}_2 M(y_2 - \tilde{C} \hat{x}) \) removes the effects of the Lipschitz nonlinearities, where matrix \( M \) and constant \( k \) will be defined hereafter. \( v_1(\tilde{e}) \) is the adaptive robust term to eliminate the effect of disturbances \( \tilde{\xi} \), defined by

\[
v_1 = -\beta \text{ sign}(s)
\]

using the sliding surface

\[
s = N \tilde{C} \tilde{e}
\]

The design of matrix \( N \) will be described hereafter. The gain adaptation algorithm is represented as follows:

\[
\dot{\beta} = \begin{cases} 
\alpha \|s\| \text{ sign}(\|s\| - \epsilon) & \text{for } \beta > \mu \\
\mu & \text{for } \beta \leq \mu 
\end{cases}
\]

with \( \beta(0) > 0, \alpha > 2, 0 < \epsilon < 1 \) and \( \mu > 0 \) being small.

**Remark 2**

In Equation (14), the parameter \( \mu \) is introduced for guaranteeing only positive value for \( \beta \). In the sequel, without loss of generality and also for the sake of clarity, only the case \( \beta > \mu \) is considered for discussion and proof.

**Lemma 3**

With the classical first-order sliding mode term (12), the adaptive gain \( \beta \), defined in (14), has an upper bound \( \beta^* \) for all \( t \geq 0 \) with \( \beta^* > \Lambda \).

The proof is similar as in [26] and is not presented here.

**Theorem 1**

For system (9), under Assumptions 1–3, consider the adaptive observer (10)–(14). If there are positive definite matrices \( P \) and \( Q \) such that

\[
\begin{align*}
\tilde{B}_2^T S^{-T} P &= M \tilde{C} \\
\tilde{\Delta}^T S^{-T} P &= N \tilde{C}
\end{align*}
\]

\[
PS^{-1}(\tilde{A}_4 - L_2 \tilde{C}) + (\tilde{A}_4 - L_2 \tilde{C})^T S^{-T} P \leq -Q
\]

and a parameter \( k \) such that

\[
k > \frac{L_2^2}{2 \lambda_{\min}(Q)}
\]

Then, the estimation error is practically stable (i.e. final and uniform boundedness). It means that all estimation error trajectories converge to a neighborhood of the origin.
Proof

Using Lemma 1, Equation (10) yields

\[
\dot{S} = S_x + L_1 y_2
\]

\[
= (A_4 - L_2 \hat{C}) \dot{z} - G y_2 + \hat{B}_1 y_1 + \hat{B}_2 \phi(C_{11}^{-1} y_1, \dot{x}_2, u) - k \hat{B}_2 M \hat{C} \hat{e} + \Delta \dot{v}_1 + L_1 \hat{C} \dot{x}
\]

From Equation (9), adding \(L_1 \hat{C} \dot{x}\) to both sides, one gets

\[
\dot{S} = \dot{A}_4 + \hat{B}_1 y_1 + \hat{B}_2 \phi(C_{11}^{-1} y_1, x_2, u) + \Delta \dot{v}_1 + L_1 \hat{C} \dot{x}
\]

The dynamic behavior of the error trajectories is

\[
\dot{\hat{e}} = S^{-1} \left((A_4 - L_2 \hat{C}) \hat{e} + \hat{B}_2 \left(\phi(C_{11}^{-1} y_1, \dot{x}_2, u) - \phi(C_{11}^{-1} y_1, x_2, u)\right) - k \hat{B}_2 M \hat{C} \hat{e} + \Delta (v_1 - \tilde{e})\right)
\]

Consider the Lyapunov function candidate

\[
V = \tilde{e}^T P \tilde{e} + \frac{1}{2}(\beta - \beta^*)^2
\]

where \(P\) is a positive definite matrix which satisfies conditions (15)–(16).

The time derivative along the state trajectories

\[
\dot{V} \leq \tilde{e}^T \left(PS^{-1}(A_4 - L_2 \hat{C}) + (A_4 - L_2 \hat{C})^T S^{-1} P\right) \tilde{e}
\]

\[
+ 2 \tilde{e}^T PS^{-1} B_2 \left(\phi(C_{11}^{-1} y_1, \dot{x}_2, u) - \phi(C_{11}^{-1} y_1, x_2, u) - k M \hat{C} \hat{e}\right)
\]

\[
+ 2 \tilde{e}^T PS^{-1} \Delta (v_1 - \tilde{e}) + (\beta - \beta^*) \alpha \| \|s\| - \epsilon\|
\]

\[
\leq -\tilde{e}^T \left(2 \tilde{e}^T Q \tilde{e} + 2 L_\phi \| \hat{e} \| M \tilde{C} \tilde{e} \right)
\]

\[
+ 2 \dot{\tilde{e}}^T N \tilde{C} \tilde{e} + 2 \| \|N \tilde{C} \tilde{e} \|\| - \beta \| \|N \tilde{C} \tilde{e} \|\| + (\beta - \beta^*) \alpha \| \|N \tilde{C} \tilde{e} \|\| - \epsilon\|
\]

If \(\|N \tilde{C} \tilde{e}\| < \epsilon\), \(\dot{V}\) is sign indefinite. Therefore, the asymptotic closed-loop stability cannot be guaranteed. Nevertheless, let us consider the case that \(\|N \tilde{C} \tilde{e}\| \geq \epsilon\). Then, it yields

\[
\dot{V} \leq -\tilde{e}^T \left(2 \tilde{e}^T Q \tilde{e} + 2 L_\phi \| \hat{e} \| M \tilde{C} \tilde{e} \right)
\]

\[
+ 2 \| \|N \tilde{C} \tilde{e} \|\| - \beta \| \|N \tilde{C} \tilde{e} \|\| + (\beta - \beta^*) \alpha \| \|N \tilde{C} \tilde{e} \|\| - \epsilon\|
\]

Because \(\alpha > 2\) and from Lemma 3, one can obtain

\[
\dot{V} \leq -\lambda_{min}(Q) \| \hat{e} \|^2 + 2 L_\phi \| \hat{e} \| M \tilde{C} \tilde{e} \|\| - 2 k \| M \tilde{C} \tilde{e} \|\|
\]

From condition (17), matrix \(\begin{bmatrix} \lambda_{min}(Q) - L_\phi & 2k \\ -L_\phi & 2k \end{bmatrix}\) > 0, it can be concluded that the estimation error trajectories converge to a neighborhood of the origin (i.e. final and uniform boundedness). Thus, we complete the proof. □

Remark 3

The convergence of the error dynamics given in Theorem 1 in the absence of measurement noise is established to a given bound around the origin, thus ensuring uniform practical stability [32]. In the presence of noise, the convergence of the estimation error can be established consequently within a small bound depending on the size of the noise and the sampling time.
Remark 4
To establish the convergence of error dynamics as shown in Theorem 1, two linear equality matrices obtained from (15) need to be solved. Similarly to [5], the solution of these linear equality matrices is equivalent to the following minimization problem. Find the smallest $\tau_1 > 0$ and $\tau_2 > 0$ such that
\[
\begin{bmatrix}
-t_1 I & \left( \hat{B}_2^T S^{-T} P - M \hat{C} \right)^T \\
0 & -I
\end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix}
-t_2 I & \left( \Delta^T S^{-T} P - N \hat{C} \right)^T \\
0 & -I
\end{bmatrix} < 0
\tag{20}
\]

Remark 5
The real estimation of the sensor fault $f_s$ is given by $f_a(t) = D^T_2 \hat{f}_s$.

3.2. Adaptive super-twisting observer for actuator fault estimation
In the following, the proposed adaptive sliding mode algorithm design will be incorporated in order to estimate the actuator fault. Let us consider subsystem (5). The following nonlinear adaptive observer can be used
\[
\dot{x}_1 = A_1 \dot{x}_1 + A_2 \dot{x}_2 + B_1 \phi(x_1, \dot{x}_2, u) + v_2
\tag{21}
\]
$v_2$ is the robust adaptive sliding mode term designed as
\[
v_2 = -k_1 \left[ \dot{x}_1 - x_1 \right]^{1/2} - k_2 \int_0^\tau \text{sign}(\dot{x}_1 - x_1) \, dt \tag{22}
\]
where the adaptive gains satisfy [27]
\[
\dot{k}_1 = \begin{cases} 
\eta_1 \text{sign} (\| \dot{x}_1 - x_1 \| - \varepsilon_1), & \text{if } k_1 > \alpha_m \\
\eta_2, & \text{if } k_1 \leq \alpha_m 
\end{cases}
\]
\[
k_2 = \eta_3 k_1 \tag{23}
\]
where $\eta_1, \eta_2, \eta_3, \varepsilon_1$ are arbitrary positive constants. The parameter $\alpha_m$ is an arbitrarily small positive constant.

Lemma 4
System (5)–(6) satisfying Assumptions 1–3 is driven to the following sliding manifold $\{ \dot{x}_1 - x_1 = \hat{x}_1 - \dot{x}_1 = 0 \}$ in finite time and remains on it.

Proof
Let us define the estimation error as
\[
e_1 = \dot{x}_1 - x_1 = \hat{x}_1 - C_{11}^1 y_1
\]
The error dynamics is as follows
\[
\dot{e}_1 = A_1 e_1 + A_2 (\hat{x}_2 - x_2) + B_1 (\phi(\hat{x}_1, \dot{x}_2, u) - \phi(x_1, x_2, u)) + v_2 - E_1 f_a
\tag{24}
\]
From [27], one can conclude that the adaptive super-twisting controller (22) ensures the establishment of a real 2-sliding mode with respect to $e_1$. \hfill \Box

Corollary 1
Provided that Assumptions 1–3 are satisfied. Consider the adaptive observer (10)–(14) and (21)–(23). Then, as $t \to \infty$, we have $\| A_2 (\hat{x}_2 - x_2) + B_1 (\phi(x_1, \dot{x}_2, u) - \phi(x_1, x_2, u)) \| \to 0$. Under such condition, the actuator fault can be reconstructed as
\[
\hat{f}_a(t) = -E_1^{-1} k_2 \int_0^t \text{sign}(\dot{x}_1 - x_1)
\tag{25}
\]
Lemma 4 implies that system (5)–(6) is driven to the sliding surface \( \{ e_1 = \dot{e}_1 = 0 \} \) in finite time and remains on it. Therefore, on sliding mode, the equivalent error dynamics of (24) becomes

\[
0 = A_2(\dot{x}_2 - x_2) + B_1(\phi(x_1, \dot{x}_2, u) - \phi(x_1, x_2, u)) - k_2 \int_0^t \text{sign} (\dot{x}_1 - x_1) \, ds - E_1 f_a \tag{26}
\]

Because \( \phi \) is Lipschitz and from Theorem 1, the following holds. As \( t \to \infty \), we have \( \| A_2 (\dot{x}_2 - x_2) + B_1(\phi(x_1, \dot{x}_2, u) - \phi(x_1, x_2, u)) \| \to 0 \). This concludes the proof. \( \square \)

4. APPLICATION TO FLEXIBLE JOINT ROBOT ARM

We consider the laboratory model of a single-link flexible joint robot [33], shown in Figure 1 and defined by the following nonlinear differential equations:

\[
\begin{align*}
\dot{\theta}_m &= \omega_m \\
\dot{\omega}_m &= \frac{k}{J_m}(\theta_l - \theta_m) - \frac{S_m}{J_m} \omega_m + \frac{K_v}{J_m} u \\
\dot{\theta}_l &= \omega_l \\
\dot{\omega}_l &= -\frac{k}{J_l}(\theta_l - \theta_m) - \frac{S_l}{J_l} \omega_l - \frac{m g b}{J_l} \sin(\theta_l)
\end{align*}
\tag{27}
\]

where \( \theta_m \) and \( \theta_l \) are the angular rotations of the motor and the link respectively, with \( \omega_m \) and \( \omega_l \) being their angular velocities, \( J_m = 0.0037 \text{ kgm}^2 \) represents the inertia of the DC motor, \( J_l = 0.093 \text{ kgm}^2 \) the inertia of the controlled link, \( k = 0.18 \text{ Nm/rad} \) the elastic constant, \( m = 0.21 \text{ kg} \) the link mass, \( g = 9.81 \text{ m/s}^2 \) the acceleration due to gravity, \( K_v = 0.08 \text{ Nm/V} \) the amplifier gain and \( S_m = 0.0083 \text{ Nm/V} \) (resp. \( S_l \)) the viscous friction coefficient of the motor (of the link resp.).

In order to evaluate the effectiveness of the proposed method, we consider the following perturbations and actuator fault for the nominal model rewritten in the following form with

\[
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix} =
\begin{bmatrix}
\theta_m \\ \omega_m \\ \theta_l \\ \omega_l
\end{bmatrix}^{T}
\]

\[
\dot{x} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
-k/J_m & -B/J_m & k/J_m & 0 \\
0 & 0 & 0 & 1 \\
k/J_l & 0 & -k/J_l & 0
\end{bmatrix} x
+ \begin{bmatrix}
0 & 0 \\
K_v/J_m & 0 \\
0 & 0 \\
0 & -m g b/J_l
\end{bmatrix} u
+ \begin{bmatrix}
e_1 \\ e_2 \\ e_3 \\ e_4
\end{bmatrix} f_a
+ \begin{bmatrix}
0 \\ 0 \\ 0 \\ -1
\end{bmatrix} \xi
\tag{28}
\]

In this example, the viscous friction coefficient \( S_l \) is assumed to be unknown. It generates the perturbation \( \xi = J_l w_l \). The actuator fault \( f_a \in \mathcal{R} \) and its distribution matrix are selected for illustration purpose and to highlight the effectiveness of the proposed strategy in reconstruction. The actuator fault distribution vector elements are chosen to be \( e_1 = 1.2, e_2 = 0.4, e_3 = 0.8, e_4 = 0.2 \). The measured output vector is

\[
y = C x + D f_x
\tag{29}
\]
where \( f_s \in \mathcal{R} \) represents the sensor fault. Here, \( C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \) and \( D = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix} \).

Hence, the dynamical system (28) can be represented in the form of (1). Setting \( \delta = 0.5 \), it is clear that Assumptions 1–2 are satisfied.

Using transformation (4), system (28) in the new coordinates \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Hy \) is described by (5)–(6) with

\[
A_1 = 0.4167, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -5.1623 \\ -0.2153 \\ 0.0918 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2.6599 & 48.6486 & 0 \\ -0.9167 & 0 & 1 \\ -0.1667 & -1.9355 & 0 \end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 21.6216 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = 1.2, \quad \Delta_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
C_{11} = 1, \quad C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}
\]

For illustration purpose, we choose

\[
f_s = \begin{cases} 
0.1 \sin(20t) & \text{if } t \leq 5 \text{ (i.e. no sensor fault)} \\
0.1 \sin(20t) + 1 \sin(5(t - 5)) & \text{if } t > 5 \text{ (i.e. sensor fault)}
\end{cases}
\]

Figure 2. (a)–(d) State estimation with the proposed sliding mode observer and state estimation error in the presence of measurement noise.
and

\[ f_a = \begin{cases} 
0.5 \sin(10t) & \text{if } t \leq 10 \\
-0.25 + 2 \sin(t - 10) & \text{if } 16.3 \leq t > 1 \\
0 & \text{if } t > 16.3
\end{cases} \]

(i.e. actuator fault)

From Lemmas 1–2, one can choose \( L_1 \) and \( L_2 \) as follows:

\[
L_1 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0.5 & 0 \\
0 & 0.5
\end{bmatrix}, \\
L_2 = \begin{bmatrix}
9.61 & 6.98 \\
-1.78 & 1.65 \\
5.10 & 12.63 \\
3.44 & 4.49 \\
3.06 & 0.53 \\
0.53 & 1.84
\end{bmatrix}
\]

From Theorem 1, one can obtain matrices \( P, Q, M, \) and \( N \) by solving the LMI conditions (16), (20) using the LMI-toolbox in Matlab. The sliding mode gains are set as \( \alpha = 1000, \epsilon = 0.0001 \) and \( \mu = 0.0001, \eta_1 = 0.5, \eta_2 = 0.4, \epsilon_1 = 0.0001, \alpha_m = 0.1 \).

The initial conditions for plant and estimator are \( x = [3 \ 2.75 \ 0.75 \ 1.5] \) and \( \hat{x} = [0 \ 0 \ 0 \ 0] \).

Recalling the transformation \( x = T^{-1} [x_1^T \ x_2^T]^T \), each component of the state \( x \) in the original coordinate is written as \( x = [x(1) \ x(2) \ x(3) \ x(4)]^T \). Similarly, each component of the estimated state \( \hat{x} = T^{-1} [\hat{x}_1^T \ \hat{x}_2^T]^T \) is denoted as \( \hat{x} = [\hat{x}(1) \ \hat{x}(2) \ \hat{x}(3) \ \hat{x}(4)]^T \). To test the robustness of the proposed design, a random noise is added to the output measurements.

Figure 2 shows the corresponding tracking performance. Despite the presence of faults and perturbation, which creates large oscillations in the robot arm motion, the observer is able to track the system state (Figure 2) with high accuracy. We see that the state estimation based on the adaptive observer (10)–(14) and (21)–(23) is satisfactory even if there are some faults.

![Figure 3.](image-url)

(a) Sensor fault and its estimate
(b) Perturbation and its estimate
(c) Actuator fault and its estimate

Figure 3. (a)–(c) Sensor fault, perturbation, and actuator fault estimations in the presence of measurement noise.
The sensor fault signal $f_s$ and the perturbation term $\xi$ are reconstructed from the adaptive observer (10)–(14) and are depicted in Figure 3(a)–(b). The actuator fault signal $f_a$ is reconstructed from the adaptive super-twisting observer (21)–(23) and is depicted in Figure 3(c).

These simulation results highlight that the proposed strategy, described in Theorem 1 and Corollary 1, manages to accomplish the objective (state and fault estimations) with good performances in terms of convergence speed, steady-state error and robustness with respect to perturbations. The proposed reconstruction method not only provides when the actuators or sensors are faulty but also gives the shape and magnitude of faults.

5. CONCLUSION

This paper considers state and fault estimations for a class of uncertain Lipschitz nonlinear systems. First, an adaptive descriptor sliding approach is proposed to jointly estimate the state and reconstruct the sensor fault while rejecting the effect of uncertainties and Lipschitz nonlinearities. Using a Lyapunov analysis, the stability condition is analyzed and the observer gains are designed such that the reduced-order system is practically stable. Then, an adaptive super-twisting observer is derived to reconstruct the actuator faults. The effectiveness of the proposed strategy is illustrated through simulations to solve the fault and state reconstruction for a robotic arm.

REFERENCES


